Regularity Theory for the Axially Symmetric Incompressible Navier-Stokes Equations Without Swirl

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Introduction

The Navier-Stokes equations are a system of partial differential equations (PDEs) describing the motion of a viscous fluid. They model a wide range of phenomena, such as air flow around a wing or the spreading of a droplet under gravity. It is thus somewhat surprising that there are many fundamental questions about this system that remain unanswered. The rigorous mathematical theory for the Navier-Stokes equations was founded by Jean Leray in his pioneering work of 1934. Since then, important contributions have been made by E. Hopf, O.A. Ladyzhenskaya and L. Caffarelli, R. Kohn and L. Nirenberg, to name but a few. However, to this date, the question of finite-time blow-up of solutions is still open.

This dissertation will give an introduction to some aspects of the mathematical theory of the Navier-Stokes equations, and describe how to analyse the regularity of solutions for the particular case of axially symmetric flow without swirl. In Chapter 1, we discuss the fundamental problem regarding the Navier-Stokes system and give a heuristic argument for why this problem is so difficult. We introduce Leray-Hopf weak solutions in Chapter 2 and prove their existence in Chapter 3 using the Galerkin method.

We will assume results from functional analysis, integration theory and the theory of Sobolev spaces, as presented in the Part B courses Banach and Hilbert Spaces, the Part A course Integration and the Part C course Functional Analytic Methods for PDEs. Some of the more frequently used results are listed in Appendix B. Notation will be introduced as required and collected in Appendix A.

Though the theory is usually presented by successively treating the stationary linear, stationary nonlinear and non-stationary nonlinear problems, we will consider the full problem from the outset. Consequently, a substantial portion of the material is dedicated to motivating the definitions and results. Nonetheless, it will not be possible to prove everything, and a list of references with details is provided.

In Chapter 4, we discuss the uniqueness and regularity of weak solutions. We derive a sufficient condition for uniqueness, and show that a unique, smooth solution exists at least for a short amount of time. A necessary condition for non-existence globally in time is then derived, namely that of finite-time blow-up. This naturally leads to local regularity theory and the famous Caffarelli-Kohn-Nirenberg Theorem, which we give a brief description of in Chapter 5. Finally, in Chapter 6 we apply this material to analyse regularity for axially symmetric flow without swirl.

1 Preliminaries

The Navier-Stokes equations governing the motion of a viscous, incompressible fluid with viscosity ν in a domain Ω are, in the absence of external forces, given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.2}$$

for any point in space-time $(x,t) \in \Omega_T \coloneqq \Omega \times (0,T)$, with $T > 0$. Throughout this dissertation, unless stated otherwise, we will assume $\Omega \subset \mathbb{R}^3$ is a bounded open set with smooth boundary $\partial\Omega$.

We impose the boundary and initial conditions

$$\boldsymbol{u}(x,t) = 0, \quad \forall\, (x,t) \in \partial\Omega \times (0,T); \tag{1.3}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \forall\, x \in \Omega. \tag{1.4}$$

Together, (1.1)-(1.4) define the *Navier-Stokes initial-boundary value problem.*

A *classical solution* is a pair $(\boldsymbol{u}, p)$ satisfying (1.1)-(1.4) pointwise, with $\boldsymbol{u}$ and p smooth enough for the equations to make sense i.e.

$$\boldsymbol{u} \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T) \quad \text{and} \quad p \in C^1(\Omega_T),$$

where

$$C^k(\Omega_T) = \left\{\, f : \Omega_T \to \mathbb{R} \mid \nabla^k f \text{ is continuous in } \Omega_T \,\right\},$$

$$C_1^2(\Omega_T) = \left\{\, f : \Omega_T \to \mathbb{R} \mid f,\, \nabla f,\, \nabla^2 f,\, \partial_t f \in C(\Omega_T) \,\right\}.$$

Here, $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is a vector valued function representing the velocity of the fluid; $p = p(x,t)$ is a scalar function representing the pressure field, and $\boldsymbol{u}_0(x)$ is a vector field vanishing on $\partial\Omega$ that satisfies $\operatorname{div} \boldsymbol{u}_0 = 0$. We use the notation

$$\partial_t \equiv \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial x_i}$$

to mean partial differentiation with respect to the time and space variables, respectively. The operators ∇, Δ and div are the usual gradient, Laplacian and divergence operators, respectively, and $\nabla^2 f$ denotes the Hessian matrix of f. The equations (1.1) and (1.2) represent conservation of linear momentum and conservation of mass, respectively (see [21] for a detailed derivation).

We do not need to prescribe initial or boundary data for the pressure, since we can use (1.1) to solve for p once $\boldsymbol{u}$ is known. Specifically, we operate with div on both sides of (1.1), interchange derivatives and use (1.2) and (1.3) to obtain p as the solution to

$$-\Delta p = \operatorname{div}((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}), \quad \text{in } \Omega;$$

$$\frac{\partial p}{\partial n} = \boldsymbol{n} \cdot (\nu \Delta \boldsymbol{u}), \qquad \text{on } \partial\Omega,$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$.

If $\nu \neq 0$, then under the transformation

$$\hat{t} = \nu^{-\frac{1}{3}} t, \quad \hat{x} = \nu^{-\frac{2}{3}} x, \quad \hat{\boldsymbol{u}} = \nu^{-\frac{1}{3}} \boldsymbol{u}, \quad \hat{p} = \nu^{-\frac{2}{3}} p,$$

the equation (1.1) takes the same form but with $\nu = 1$. Thus, we may assume that $\nu = 1$. In Cartesian coordinates (x_1, x_2, x_3), the equations in component form then read

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} - \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \frac{\partial p}{\partial x_i} = 0, \tag{1.5}$$

$$\frac{\partial u_k}{\partial x_k} = 0, \tag{1.6}$$

for each $i = 1, 2, 3$. Here we are employing the summation convention, where a repeated free index in the same term implies summation over all values of that index.

The fundamental question related to these equations is that of global existence in time and uniqueness of a classical solution $(\boldsymbol{u}, p)$. One might ask why this question is important, and indeed why we should expect it to be true. Since these equations model physical phenomena, we should expect that smooth initial conditions give rise to a unique, smooth velocity. A confirmation of this would provide solid evidence towards the validity of the model, while a counterexample would suggest that the model might need modification.

Moreover, this question is related to the problem of *turbulence*, described by Feynman [7] as "the most important unsolved problem of classical physics". A resolution of the problem might come with advances in understanding turbulence, which would have huge implications in mathematical modelling. On the other hand, this is an important mathematical problem in its own right, and a solution will likely involve new ideas and interesting mathematics.

However, we can already see a few difficulties in attempting to solve this problem. Firstly, this is a set of four coupled equations, and there is no obvious way of constructing a solution. Indeed, an explicit solution is known only in a few special cases. Furthermore, as Galdi [10] points out, the variables $\boldsymbol{u}$ and p do not appear in a "symmetric way" in the equations since (1.2) is not of the form

$$\frac{\partial p}{\partial t} = F(\boldsymbol{u}, p).$$

Thus, the system does not belong to one of the classical categories of second-order PDEs, though it is similar to a quasi-linear parabolic equation. The theory for the classical categories of PDEs is well studied, and the problem obtained by dropping the pressure term and the divergence-free condition can be solved completely (see Galdi [10] §1). The solvability of this simpler problem is not much help to us since it is a fundamentally different problem from (1.1)-(1.4), primarily because the energy is of a different form.

On the other hand, the problem obtained by dropping the nonlinear term $(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u}$ can also be completely solved, as we will show in Chapters 3 and 4. Thus, it is the combination of the lack of symmetry and the presence of the nonlinear term that makes this problem so difficult.

The equation (1.1) describes primarily a conflict between the nonlinear term $(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u}$, and the stabilising dissipative term $\Delta\boldsymbol{u}$. If the dissipative term dominates the nonlinear one then, as we said, we can expect existence and uniqueness of a smooth solution. Conversely, if the nonlinear term is the dominating term, then we might expect turbulent behaviour. However, existence and uniqueness results are known for many other nonlinear PDEs, the two-dimensional Navier-Stokes problem among them, so what is the issue here?

As we shall see, the problem reduces to the question of smoothness of solutions. To see the issue, we follow Tao [28] and perform some dimensional analysis. Consider a parcel of fluid with typical length L, rotating at a typical velocity U. The nonlinear and dissipative terms scale like U^2/L and U/L^2, respectively. Hence, we can expect linear behaviour if $U \ll 1/L$, and nonlinear behaviour if $U \gg 1/L$.

The most obvious way of controlling the behaviour of the velocity is through the kinetic energy of the fluid:

$$E(t) = \frac{1}{2}\rho \int_{\Omega} |\boldsymbol{u}(x,t)|^2 \,\mathrm{d}x.$$

Since the system is *dissipative*, $E(t) \leq E(0) = O(1)$, $\forall\, t \geq 0$. For our parcel of fluid, the kinetic energy scales like $E \sim U^2 L^3$, and we thus have an upper bound $U = O(L^{-3/2})$. Note that this is consistent with the dominant term being the nonlinear term, as $L \to 0$. Hence, if the length of this parcel is made smaller and smaller and the kinetic energy is "squeezed" into this parcel, the behaviour is highly nonlinear.

So suppose that at $t = 0$ all the kinetic energy is concentrated into this parcel i.e. $U \approx L^{-3/2}$. Reduce the parcel to a size $L/2$, so that it is rotating with velocity $U \sim (L/2)^{-3/2}$. Since time scales like L/U, this happens in a time $t \sim L^{5/2}$. Using this argument n times, we find a parcel of size $L/2^n$, rotating with velocity

$U \sim (L/2^n)^{-3/2}$, at a time

$$t \sim L^{5/2} + \left(\frac{L}{2}\right)^{5/2} + \left(\frac{L}{2^2}\right)^{5/2} + ... + \left(\frac{L}{2^n}\right)^{5/2}$$

$$= L^{5/2} \sum_{k=0}^{n} 2^{-5k/2}.$$

Sending $n \to \infty$, we find that the velocity $U \to \infty$ in a finite time. Indeed, Tao [29] used this idea to demonstrate finite-time blow-up for an averaged form of the 3D Navier-Stokes equations.

The reason why this works is because at each iteration, the dissipative term is negligible compared to the nonlinear term, so we can ignore the dissipation of energy due to viscosity. The total cumulative dissipation of energy is

$$\phi(t) = \frac{1}{2} \int_0^t \int_\Omega |\nabla \boldsymbol{u}(x, \tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau.$$

When we reduce the size of the parcel we have that $\phi \sim L^{1/2} \ll 1 \sim E$ as $L \to 0$. This also explains why this argument cannot work in two dimensions. Here, the kinetic energy scales like $E \sim U^2 L^2$, and hence the upper bound for the velocity is $U = O(L^{-1})$. This is precisely the situation where the nonlinear and dissipative forces balance, and we therefore cannot neglect the viscous dissipation of energy.

2 Weak Solutions

A key idea we will need is the concept of a *weak solution*, introduced by Leray [20]. The idea is to generalise the notion of a solution to a wider class of functions with the expectation that the weaker requirements on the smoothness of solutions allows one to more easily establish existence. We then hope that in fact a weak solution is actually a classical solution, leading to the question of *regularity* of weak solutions. Obviously, for this to work, any classical solution must be a weak solution.

Our goal in this chapter is to define *Leray-Hopf weak solutions*, and we will do this with the following two ideas in mind:

1. A weak solution will have less regularity than a classical solution. Consequently, we must derive conditions analogous to (1.1)-(1.4) that appropriately capture the equations of motion, initial condition, boundary data, and energy conservation, but with fewer assumptions regarding smoothness.

2. These conditions should be sufficiently stringent so as to ensure that any function with enough regularity that does satisfy the conditions is a classical solution.

Since we need a classical solution to also be a weak solution, we will motivate the definition of a weak solution by assuming that $\boldsymbol{u}(x,t)$ is a classical solution, so $\boldsymbol{u}$ is in particular square integrable, and then deriving conditions that $\boldsymbol{u}$ must satisfy.

Introduce $L^p(\Omega)$, the Banach space of of real-valued, measurable functions satisfying

$$\|f\|_{p,\Omega} := \left(\int_\Omega |f(x)|^p \, \mathrm{d}x\right)^{1/p} < \infty, \quad \text{if } p \in [1,\infty);$$

$$\|f\|_{\infty,\Omega} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{\, M \in \mathbb{R} \mid |f(x)| \le M \text{ a.e. in } \Omega \,\}, \quad \text{if } p = \infty.$$

In particular, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(f,g) = \int_\Omega f(x)g(x)\,\mathrm{d}x.$$

When we say that a vector valued function $\boldsymbol{f}$ belongs to a function space X, we will mean that each component f_i belongs to X, and

$$\|\boldsymbol{f}\|_X = \|\,|\boldsymbol{f}|\,\|_X = \left\| \left(\sum_i f_i^2\right)^{1/2} \right\|_X .$$

Recall now *Hölder's inequality*: if $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$, with $1 \le p,\, p' \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|fg\|_{1,\Omega} \le \|f\|_{p,\Omega} \cdot \|g\|_{p',\Omega} \,. \tag{2.1}$$

The case $p = p' = 2$ is known as the *Cauchy-Schwarz inequality.*

By induction, we obtain the generalized Hölder inequality: if $f_i \in L^{p_i}(\Omega)$ with $1 \le p_1 < \ldots < p_n \le \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = 1$, then

$$\left\| \prod_{i=1}^{n} f_i \right\|_{1,\Omega} \le \prod_{i=1}^{n} \|f_i\|_{p_i,\Omega} \,. \tag{2.2}$$

2.1 Equations of Motion

Assume $\boldsymbol{u}(x,t)$ and $p(x,t)$ are *smooth* i.e. infinitely differentiable with respect to x and t in Ω_T. This space of functions is denoted by $C^\infty(\Omega_T)$. Multiply the momentum equation (1.5) by φ_i, where $\boldsymbol{\varphi} \in C^\infty_{0,0}(\Omega_T)$, and

$$C^\infty_{0,0}(\Omega_T) = \{\, \boldsymbol{\varphi} \in C^\infty_0(\Omega_T) \mid \operatorname{div} \boldsymbol{\varphi} = 0 \,\} \,,$$

$$C^\infty_0(\Omega_T) = \{\, f \in C^\infty(\Omega_T) \mid \exists\, K \subset \Omega_T \text{ compact such that } f = 0 \text{ outside } K \,\} \,.$$

Since $\boldsymbol{u}, \boldsymbol{\varphi} \in L^2(\Omega_T)$, we can integrate over Ω_T. Applying integration by parts, we obtain the identities

$$\iint_{\Omega_T} \frac{\partial u_i}{\partial t} \varphi_i \,\mathrm{d}x\,\mathrm{d}t = -\iint_{\Omega_T} u_i \frac{\partial \varphi_i}{\partial t} \,\mathrm{d}x\,\mathrm{d}t,$$

$$\iint_{\Omega_T} \frac{\partial p}{\partial x_i} \varphi_i \,\mathrm{d}x\,\mathrm{d}t = 0,$$

$$\iint_{\Omega_T} \frac{\partial^2 u_i}{\partial x_k \partial x_k} \varphi_i \,\mathrm{d}x\,\mathrm{d}t = -\iint_{\Omega_T} \frac{\partial u_i}{\partial x_k} \frac{\partial \varphi_i}{\partial x_k} \,\mathrm{d}x\,\mathrm{d}t.$$

Thus, $\boldsymbol{u}(x,t)$ satisfies

$$-\iint_{\Omega_T} u_i \frac{\partial \varphi_i}{\partial t} \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} u_k \frac{\partial u_i}{\partial x_k} \varphi_i \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} \frac{\partial u_i}{\partial x_k} \frac{\partial \varphi_i}{\partial x_k} \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.3}$$

for every $\boldsymbol{\varphi} \in C^\infty_{0,0}(\Omega_T)$. Notice that the pressure does not appear in this equation.

Furthermore, if $\phi(x,t) \in C^\infty_0(\Omega_T)$, then (1.6) becomes

$$\iint_{\Omega_T} u_i \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.4}$$

If the assumption that $\boldsymbol{u}$ is smooth is now dropped, (2.3) only makes sense if we assume some differentiability properties of $\boldsymbol{u}$. Specifically, for each $t \in [0,T]$, the function $\boldsymbol{u}(\cdot,t)$ must belong to the *Sobolev space* $W^{1,2}(\Omega)$ consisting of those functions in $L^2(\Omega)$ for which the weak spatial derivatives exist for each component and are in $L^2(\Omega)$. The norm on $W^{1,2}(\Omega)$ is

$$\|f\|_{1,2,\Omega} \coloneqq \|f\|_{2,\Omega} + \|\nabla f\|_{2,\Omega}\,.$$

It is not immediately clear what kind of integrability we need in time. Indeed, it will be much easier to determine this from the energy equation (cf. §2.4). Drop the time integral in (2.3). Then, by the Cauchy-Schwarz inequality, the quadratic terms make sense. For the cubic-like term (cf. Lemma 3.2), we may use the generalized Hölder inequality (2.2) and Sobolev embedding (B.10) to obtain

$$\begin{aligned}\left|\int_\Omega u_k \frac{\partial u_i}{\partial x_k}\varphi_i \,\mathrm{d}x\right| &\leq \|u_k\|_{6,\Omega}\cdot\|\varphi_i\|_{3,\Omega}\cdot\|\nabla u_i\|_{2,\Omega}\\ &\leq C\|\boldsymbol{u}\|_{1,2,\Omega}\cdot\|\boldsymbol{\varphi}\|_{1,2,\Omega}\cdot\|\boldsymbol{u}\|_{1,2,\Omega} < \infty.\end{aligned}$$

Thus, the equation

$$\int_\Omega u_i \frac{\partial \varphi_i}{\partial t}\,\mathrm{d}x + \int_\Omega u_k \frac{\partial u_i}{\partial x_k}\varphi_i\,\mathrm{d}x + \int_\Omega \frac{\partial u_i}{\partial x_k}\frac{\partial \varphi_i}{\partial x_k}\,\mathrm{d}x \;= 0$$

makes sense for each $t \in [0,T]$.

Assume now that $\boldsymbol{u}$ is sufficiently smooth and satisfies (2.3) and (2.4). By considering those $\phi(x,t) = \gamma(t)\psi(x)$, where $\gamma \in C_0^\infty((0,T))$ and $\psi \in C_0^\infty(\Omega)$, and applying the Fubini-Tonelli theorem (B.2) and the fundamental lemma of calculus of variations (B.7), we reduce (2.4) to

$$\int_\Omega u_i \frac{\partial \psi}{\partial x_i}\,\mathrm{d}x = 0$$

for every $\psi \in C_0^\infty(\Omega)$. By the divergence theorem, the divergence free condition (1.6) holds.

Similarly, consider $\boldsymbol{\varphi} \in C_{0,0}^\infty(\Omega_T)$ such that $\boldsymbol{\varphi}(x,t) = \gamma(t)\boldsymbol{\psi}(x)$, with $\gamma \in C_0^\infty((0,T))$ and $\boldsymbol{\psi} \in C_{0,0}^\infty(\Omega)$. Then

$$\int_\Omega \psi_i \left\{\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} - \frac{\partial^2 u_i}{\partial x_k \partial x_k}\right\}\mathrm{d}x = 0. \tag{2.5}$$

By choosing $\boldsymbol{\psi} = \nabla\times\boldsymbol{\Psi} \in C_{0,0}^\infty(\Omega)$, and applying the identity

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}),$$

we find that

$$\int_\Omega \boldsymbol{\Psi} \cdot \nabla \times \left\{ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} \right\} \mathrm{d}x = 0, \quad \forall\, \boldsymbol{\Psi} \in C_0^\infty(\Omega).$$

Then, by Stokes' theorem, there exists a function p such that (1.5) is satisfied. Thus, a sufficiently smooth $\boldsymbol{u}(x,t)$ satisfying (2.3) and (2.4) yields the existence of a function $p(x,t)$ such that the equations of motion (1.1) and (1.2) are satisfied.

One might think that the above shows that we only need to consider $\boldsymbol{u}$ when dealing with the existence of weak solutions, since the existence of a corresponding p follows immediately from the above argument. However, we have to be careful here; in general we do not know *a priori* that the term in the braces in (2.5) is continuously differentiable. Instead, if $\boldsymbol{u}$ is sufficiently smooth to allow (2.5) to hold, we can apply the following result due to Ladyzhenskaya [17].

Theorem 2.1. *Let*

$$G(\Omega) := \left\{ \boldsymbol{v} \in L^2(\Omega) \mid \boldsymbol{v} = \nabla p \text{ for some } p \in L^2(\Omega) \right\},$$
$$H(\Omega) := \left\{ \boldsymbol{v} \in L^2(\Omega) \mid \exists\, \boldsymbol{\psi}_n \in C_{0,0}^\infty(\Omega) \text{ such that } \|\boldsymbol{v} - \boldsymbol{\psi}_n\|_{2,\Omega} \to 0 \right\}.$$

Then
$$L^2(\Omega) = G(\Omega) \oplus H(\Omega),$$

so any $\boldsymbol{w} \in L^2(\Omega)$ can be uniquely decomposed as $\boldsymbol{w} = \boldsymbol{u} + \nabla p$ with $\boldsymbol{u} \in H(\Omega)$, $p \in L^2(\Omega)$.

Here, we have introduced (cf. §2.4) the "energy space" $H = H(\Omega)$, the closure of $C_{0,0}^\infty(\Omega)$ in $L^2(\Omega)$. This is an important class of functions that plays a fundamental role in the theory of the Navier-Stokes equations. We omit the proof of Theorem 2.1, and refer the reader to Seregin [25] Theorem 7.11, Temam [30] Proposition I.1.1, Galdi [9] §III.1 and Ladyzhenskaya [17] §I.2 for the details.

By continuity, (2.5) must hold for every $\boldsymbol{\psi} \in H(\Omega)$. Theorem 2.1 then implies that the term in the braces is the gradient of some function p and (1.5) is satisfied. The problem of finding a solution pair $(\boldsymbol{u}, p)$ is then reduced to finding only $\boldsymbol{u}$.

2.2 Boundary Condition

We now derive a characterization of the boundary condition (1.3) in terms of test functions. Since $\boldsymbol{u}(\cdot,t) \in W^{1,2}(\Omega)$ only, pointwise properties do not necessarily make sense. Suppose then that $\boldsymbol{u}(\cdot,t)$ is the limit of functions in $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. Denote this space by

$$W_0^{1,2}(\Omega) = \left\{ f \in W^{1,2}(\Omega) \mid \exists\, f_n \in C_0^\infty(\Omega) \ \text{ s.t. } \ \|f - f_n\|_{1,2,\Omega} \to 0 \right\}.$$

Recall that we also need $\boldsymbol{u}$ to be divergence free. Since the weak spatial derivatives of $\boldsymbol{u}(\cdot,t)$ exist, we replace (2.4) with the condition that $\boldsymbol{u}(\cdot,t)$ is the limit of functions in $C_{0,0}^{\infty}(\Omega)$ in $W^{1,2}(\Omega)$. We denote this space by

$$V = V(\Omega) := \left\{ \boldsymbol{u} \in W^{1,2}(\Omega) \mid \exists\, \boldsymbol{\varphi}_n \in C_{0,0}^{\infty}(\Omega) \text{ s.t. } \|\boldsymbol{u} - \boldsymbol{\varphi}_n\|_{1,2,\Omega} \to 0 \right\}.$$

This, like H, is an "energy space", and is one of the fundamental classes of functions that we shall be working in.

Since Ω is bounded, Poincaré's inequality (B.9) implies that the norms $\|\cdot\|_{1,2,\Omega}$ and $\|\nabla(\,\cdot\,)\|_{2,\Omega}$ are equivalent for V. We may thus equip V with the norm $\|\nabla(\,\cdot\,)\|_{2,\Omega}$ with respect to which it is a Hilbert space. We now introduce the trace operator in the form of the following lemma.

Lemma 2.2. *There exists a bounded linear operator*

$$\gamma : W^{1,2}(\Omega) \to L^2(\Omega)$$

such that:

(i) $\gamma(f) = f\,|_{\partial\Omega}\,, \ \ \forall\, f \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.

(ii) $\|\gamma(f)\|_{2,\Omega} \le C\|f\|_{1,2,\Omega}\,, \ \ \forall\, f \in W^{1,2}(\Omega)$, *where C is a constant.*

(iii) $f \in W_0^{1,2}(\Omega) \iff \gamma(f) = 0$.

This is a classic result in the theory of Sobolev spaces. For a proof, see Evans [4] §5.5.

Thus, the boundary condition is satisfied in a weak sense if $\boldsymbol{u}(\cdot,t) \in V \subset W_0^{1,2}(\Omega)$. Obviously, if $\boldsymbol{u}(\cdot,t)$ is a classical solution then $\boldsymbol{u}(\cdot,t) \in V$. Conversely, if $\boldsymbol{u}(\cdot,t)$ is sufficiently smooth and satisfies $\gamma(\boldsymbol{u}(\cdot,t)) = 0$, then the above lemma shows that $\boldsymbol{u}(\cdot,t)\,|_{\partial\Omega} = 0$, as required.

2.3 Initial Condition

We impose a weak version of the initial condition (1.4) by requiring that

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0(\cdot)\|_{2,\Omega} \to 0 \quad \text{as} \quad t \to 0^+. \tag{2.6}$$

It is clear that a classical solution must satisfy (2.6). Conversely, if $\boldsymbol{u}(x,t)$ is sufficiently smooth, then (2.6) implies that $\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)$ for almost all $x \in \Omega$, and we conclude from continuity of $\boldsymbol{u}(x,t)$ that $\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)$ in Ω.

However, we will see[1] that imposing this condition is actually unnecessary, since it will follow from energy conservation and the following weak notion of continuity in

[1] cf. Remark 2.3.

time. We impose that the mapping $S_w : [0,T] \to \mathbb{R}$ is continuous on $[0,T]$ for each $\boldsymbol{w} \in L^2(\Omega)$, where

$$S_w \,:\, t \mapsto \int_\Omega \boldsymbol{u}(x,t) \cdot \boldsymbol{w}(x)\,\mathrm{d}x. \tag{2.7}$$

It is clear that if $\boldsymbol{u}$ is continuous in time, then this holds.

2.4 Energy Inequality

Suppose $\boldsymbol{u}$ is a smooth solution to (1.5) and (1.6). Multiply (1.5) by u_i, sum over $i = 1, 2, 3$, and integrate over Ω:

$$\int_\Omega u_i \frac{\partial u_i}{\partial t}\,\mathrm{d}x + \int_\Omega u_i \frac{\partial u_i}{\partial x_k} u_k\,\mathrm{d}x - \int_\Omega u_i \frac{\partial^2 u_i}{\partial x_k \partial x_k}\,\mathrm{d}x + \int_\Omega u_i \frac{\partial p}{\partial x_i}\,\mathrm{d}x = 0. \tag{2.8}$$

The first, third and fourth terms in (2.8) may be written as

$$\int_\Omega u_i \frac{\partial u_i}{\partial t}\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_i^2\,\mathrm{d}x, \tag{2.9}$$

$$-\int_\Omega u_i \frac{\partial^2 u_i}{\partial x_k \partial x_k}\,\mathrm{d}x = \int_\Omega \left(\frac{\partial u_i}{\partial x_k}\right)^2\,\mathrm{d}x \;-\; \int_\Omega \frac{\partial}{\partial x_k}\left(u_i \frac{\partial u_i}{\partial x_k}\right)\,\mathrm{d}x, \tag{2.10}$$

$$\int_\Omega u_i \frac{\partial p}{\partial x_i}\,\mathrm{d}x = \int_\Omega \frac{\partial}{\partial x_i}(p u_i)\,\mathrm{d}x \;-\; \int_\Omega p \frac{\partial u_i}{\partial x_i}\,\mathrm{d}x, \tag{2.11}$$

respectively. The second term in (2.8) vanishes since

$$\int_\Omega u_i \frac{\partial u_i}{\partial x_k} u_k\,\mathrm{d}x = \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_k}(u_k u_i^2)\,\mathrm{d}x \;-\; \underbrace{\frac{1}{2}\int_\Omega u_i^2 \operatorname{div}\boldsymbol{u}\,\mathrm{d}x}_{=0}$$

$$= \frac{1}{2}\int_\Omega \operatorname{div}(u_i^2\,\boldsymbol{u})\,\mathrm{d}x \;=\; \frac{1}{2}\int_{\partial\Omega} u_i^2\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s \;=\; 0.$$

By the divergence theorem and the zero boundary condition, (2.10) is

$$\int_\Omega |\nabla \boldsymbol{u}|^2\,\mathrm{d}x \;-\; \int_\Omega \operatorname{div}(\boldsymbol{u}^T \nabla \boldsymbol{u})\,\mathrm{d}x = \int_\Omega |\nabla \boldsymbol{u}|^2\,\mathrm{d}x.$$

Similarly, (2.11) is identically zero. Thus, we are left with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega |\nabla \boldsymbol{u}|^2\,\mathrm{d}x = 0.$$

Integrating with respect to time and applying the initial condition, we obtain the *energy equality*:

$$\frac{1}{2}\int_{\Omega}|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x+\int_0^t\int_{\Omega}|\nabla\boldsymbol{u}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau=\frac{1}{2}\int_{\Omega}|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x, \tag{2.12}$$

for every $t\in[0,T]$. The first term represents the total kinetic energy of the fluid at time t, while the second term represents the total dissipation of energy due to viscosity over the time period $[0,t]$. By the law of conservation of energy, this must equal the initial kinetic energy.

However, it turns out that imposing that a weak solution satisfies the energy equality is too restrictive, essentially because we can only prove weak convergence, rather than strong convergence (cf. §3.1). Thus, we instead impose that a weak solution $\boldsymbol{u}$ must satisfy an *energy inequality*:

$$\frac{1}{2}\int_{\Omega}|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x+\int_0^t\int_{\Omega}|\nabla\boldsymbol{u}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau\le\frac{1}{2}\int_{\Omega}|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x, \tag{2.13}$$

for every $t\in[0,T]$. It is clear now why we have emphasized that $\boldsymbol{u}(\cdot,t)\in L^2(\Omega)$; the presence of the quadratic terms in the energy equation suggests that the problem should be formulated on a Hilbert space.

Remark 2.3. *Condition* (2.6) *follows from the energy inequality* (2.13) *and the weak continuity in time* (2.7). *Indeed,* (2.7) *implies that* $\boldsymbol{u}(\cdot,t)\rightharpoonup\boldsymbol{u}_0$ *in* $L^2(\Omega)$ *as* $t\to 0^+$ *i.e.* $\boldsymbol{u}$ *converges weakly to* $\boldsymbol{u}_0$ *in* $L^2(\Omega)$. *By the Banach-Steinhaus theorem, the sequence* $\boldsymbol{u}(\cdot,t)$ *is norm bounded, and* $\liminf_{t\to 0^+}\|\boldsymbol{u}(\cdot,t)\|_2$ *exists. Note that*

$$\|\boldsymbol{u}_0\|_{2,\Omega}^2=|(\boldsymbol{u}_0\,,\boldsymbol{u}_0)|=\liminf_{t\to 0^+}|(\boldsymbol{u}_0\,,\boldsymbol{u}(\cdot\,,t))|\le\|\boldsymbol{u}_0\|_{2,\Omega}\liminf_{t\to 0^+}\|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}$$

$$\implies\liminf_{t\to 0^+}\|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}\ge\|\boldsymbol{u}_0\|_{2,\Omega}\,.$$

On the other hand, (2.13) *implies*

$$\limsup_{t\to 0^+}\|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}\le\|\boldsymbol{u}_0\|_{2,\Omega}.$$

This means that

$$\lim_{t\to 0^+}\|\boldsymbol{u}(\cdot\,,t)\|_{2,\Omega}=\|\boldsymbol{u}_0\|_{2,\Omega}.$$

Since weak convergence and norm convergence in a Hilbert space together imply strong convergence, (2.6) *holds.*

From the energy inequality (2.13), we see that the kinetic energy $\|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}^2$ must be bounded as a function of time. This leads us to consider the function $\boldsymbol{u}(x,t)$ as a mapping of time t into some Banach space $X=X(\Omega)$, by associating with $\boldsymbol{u}(x,t):\Omega\times(0,T)$ the mapping $\boldsymbol{u}:(0,T)\to X$.

We thus introduce the space $L^p(0,T\,;X)$, with the norm

$$\|\boldsymbol{u}\|_{L^p(0,T\,;X)} = \left(\int_0^T \|\boldsymbol{u}(\cdot,t)\|_X \,\mathrm{d}t\right)^{1/p}, \qquad \text{if } 1 \le p < \infty,$$

$$\|\boldsymbol{u}\|_{L^\infty(0,T\,;X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|\boldsymbol{u}(\cdot,t)\|_X\,, \qquad \text{if } p = \infty.$$

These are known as *Lebesgue-Bochner* spaces. It is well known that many properties that hold for $L^p(\Omega)$ also hold for $L^p(0,T\,;X)$. For example, the dominated convergence theorem, Minkowski's inequality and Hölder's inequality are all true. Consequently, completeness of $L^p(0,T\,;X)$ can be proved using similar arguments to the ones used to prove completeness of $L^p(\mathbb{R})$.

Recall the space H introduced in Theorem 2.1. We need $\boldsymbol{u}$ to be bounded as a function of time into this space, so we impose that $\boldsymbol{u} \in L^\infty(0,T\,;H)$, with the norm

$$\|\boldsymbol{u}\|_{L^\infty(0,T;H)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}.$$

Similarly, we also need $\|\nabla\boldsymbol{u}(\cdot,t)\|_{2,\Omega}$ to be in $L^2(0,T)$. Accordingly, we impose that $\boldsymbol{u} \in L^2(0,T\,;V)$, with the norm

$$\|\boldsymbol{u}\|_{L^2(0,T;V)} = \left(\int_0^T \|\nabla\boldsymbol{u}(\cdot,t)\|_{2,\Omega}^2 \,\mathrm{d}t\right)^{1/2}.$$

Returning to the momentum equation (2.3), we now see that it makes sense if $\boldsymbol{u} \in L^\infty(0,T\,;H) \cap L^2(0,T\,;V)$.

2.5 Leray-Hopf Weak Solutions

Definition 2.4. Let $\boldsymbol{u}_0 \in H$. A function $\boldsymbol{u}$ is called a *Leray-Hopf weak solution* to the initial-boundary value problem (1.1)-(1.4) if it satisfies:

1. $\boldsymbol{u} \in L^\infty(0,T\,;H) \cap L^2(0,T\,;V)$.
2. $S_w : t \mapsto \int_\Omega \boldsymbol{u}(x,t)\cdot\boldsymbol{w}(x)\,\mathrm{d}x$ is continuous on $[0,T]$ for each $\boldsymbol{w} \in L^2(\Omega)$.
3. For every $\boldsymbol{\varphi} \in C^\infty_{0,0}(\Omega_T)$,
$$-\iint_{\Omega_T} u_i \frac{\partial\varphi_i}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} u_k \frac{\partial u_i}{\partial x_k}\varphi_i\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} \frac{\partial u_i}{\partial x_k}\frac{\partial\varphi_i}{\partial x_k}\,\mathrm{d}x\,\mathrm{d}t = 0.$$
4. $\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}_0(\cdot)\|_2 \to 0 \quad \text{as} \quad t \to 0^+$;
5. For every $t \in [0,T]$ the weak energy inequality is satisfied:
$$\frac{1}{2}\int_\Omega |\boldsymbol{u}(x,t)|^2\,\mathrm{d}x + \int_0^t \int_\Omega |\nabla\boldsymbol{u}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{1}{2}\int_\Omega |\boldsymbol{u}_0(x)|^2\,\mathrm{d}x.$$

The definition of weak solutions is somewhat arbitrary in the sense that one could define a "weak solution" in a different way, as long as it incorporated the equations of motion, initial condition, boundary data and energy equality in an appropriate way. The point is that this definition is the one that allows us to prove the existence theorem (relatively) easily. However, as we shall see in Chapter 4, proving uniqueness for this class of solutions is extremely difficult. In this sense, the definition needs to be "weak" enough to prove existence, but also "strong" enough to prove uniqueness.

3 Existence

In this chapter, we will state and prove an existence theorem regarding weak solutions. The case $\Omega = \mathbb{R}^3$ was proved by Leray [20], while the case $\Omega \subset \mathbb{R}^3$ bounded was proved by Hopf [13].

Theorem 3.1 (Existence of weak solutions)**.**
There exists a Leray-Hopf weak solution to the initial-boundary value problem (1.1)-(1.4).

Before we proceed with a proof, we give an equivalent formulation of the equation of motion (2.3). Introduce the operator b on the space $V \times V \times V$, defined by

$$b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{w}) = \int_\Omega u_k \frac{\partial v_i}{\partial x_k} w_i \,\mathrm{d}x = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\,,\boldsymbol{w}). \tag{3.1}$$

Lemma 3.2.

(i) b is a trilinear continuous form.

(ii) $b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{w}) = -b(\boldsymbol{u}\,,\boldsymbol{w}\,,\boldsymbol{v}), \quad \forall\, \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in V.$

(iii) $b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{v}) = 0, \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in V.$

Proof.
(i) Linearity is clear. For continuity, the generalized Hölder inequality (2.2), Sobolev embedding (B.10) and Poincaré's inequality (B.9) imply that

$$|b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{w})| \le \|\boldsymbol{u}\|_{4,\Omega}\cdot\|\nabla\boldsymbol{v}\|_{2,\Omega}\cdot\|\boldsymbol{w}\|_{4,\Omega} \le C\,\|\nabla\boldsymbol{u}\|_{2,\Omega}\cdot\|\nabla\boldsymbol{v}\|_{2,\Omega}\cdot\|\nabla\boldsymbol{w}\|_{2,\Omega}\,,$$

for every $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in V$.

(ii) By continuity, it is enough to prove this for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in C_{0,0}^\infty(\Omega)$. By integration by parts,

$$\begin{aligned} b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{w}) &= \int_\Omega u_k \frac{\partial v_i}{\partial x_k} w_i \,\mathrm{d}x \\ &= \int_\Omega \frac{\partial}{\partial x_k}(u_k\, v_i\, w_i)\,\mathrm{d}x - \int_\Omega \frac{\partial u_k}{\partial x_k} v_i w_i \,\mathrm{d}x - \int_\Omega u_k \frac{\partial w_i}{\partial x_k} v_i \,\mathrm{d}x. \end{aligned}$$

The first integral is zero by the divergence theorem and the boundary conditions; the second integral is zero since $\operatorname{div}\boldsymbol{u} = 0$; the last integral is $-b(\boldsymbol{u}\,,\boldsymbol{w}\,,\boldsymbol{v})$. Note that (iii) is an immediate consequence of (ii). □

Introduce the dual spaces of V and H, denoted by V' and H'. Note that the inclusion operator $i : V \hookrightarrow H$ is linear and continuous by Poincaré's inequality (B.9), that $i(V) = V \subset H$ and $\overline{i(V)} = \overline{V} = H$.

Lemma 3.3. *[Folland [8] Exercise 5.22.] Let X and Y be normed spaces with dual spaces X' and Y', respectively. Let $T \in \mathcal{B}(X, Y)$ a bounded linear operator and $T^* \in \mathcal{B}(Y', X')$ be the adjoint operator. Then:*

(i) T^* is injective $\iff$ $\overline{T(X)} = Y$;

(ii) If $\overline{T^*(Y')} = X'$, then T is injective. The converse is true if X is reflexive.

Proof.
(i) Suppose $\overline{T(X)} \neq Y$, so that there exists $y \in Y \setminus \overline{T(X)}$. Since $y \neq 0$, by the Hahn-Banach theorem (B.3) there exists $f \in Y'$ such that $f(y) \neq 0$ but $f(z) = 0$ for all $z \in \overline{T(X)}$. Thus, $T^*f = f \circ T = 0$, so T^* is not injective.

Conversely, suppose T^* is not injective, so that there exists $f \in Y'$ such that $f \neq 0$ but $T^*f = 0$. Then there exists $y \in Y$ such that $f(y) \neq 0$. By continuity, there exists $\delta > 0$ such that $f(y') \neq 0$ for any $y' \in B(y, \delta)$, where

$$B(y, \delta) = \{\, y' \in Y \mid \|y - y'\|_Y < \delta \,\}.$$

Clearly, $B(y, \delta) \notin T(X)$, and $\overline{T(X)} \subset Y \backslash B(y, \delta) \subsetneq Y$.

(ii) Suppose T is not injective, so $Tx = 0$ with $x \neq 0$ and $\|x\|_X = 1$. By the Hahn-Banach theorem (B.3), there exists $f \in X'$ such that

$$f(x) = \|x\|_X, \quad \|f\|_{X'} = 1.$$

We claim now that $B_{X'}(f, 1) \notin \overline{T^*(Y')}$, which then proves that $\overline{T^*(Y')} \neq X'$. Indeed, if $g \in B_{X'}(f, 1)$, then

$$|g(x) - 1| = |g(x) - f(x)| \leq \|g - f\|_{X'} \|x\|_X < 1,$$

so that $g(x) \neq 0$, and hence $g \notin \overline{T^*(Y')}$.

Conversely, suppose X is reflexive and $\overline{T^*(Y')} \neq X'$. There exists $f \in X' \setminus \overline{T^*(Y')}$, and hence $\hat{x} \in X''$ such that $\hat{x}(f) \neq 0$ but $\hat{x}(g) = 0$ for every $g \in \overline{T^*(Y')}$. By reflexivity, $f(x) \neq 0$, so that $x \neq 0$, and $g(x) = 0$ for every $g \in \overline{T^*(Y')}$. If we can show that $Tx = 0$, then the result follows. But if $Tx \neq 0$, then by the Hahn-Banach theorem (B.3) there exists $h \in Y'$ such that $g(x) := h(Tx) \neq 0$, which is the required contradiction. □

Following Temam [30] §III.1, we argue as follows. Set $X = V$, $Y = H$, and $T = i : V \hookrightarrow H$ in Lemma 3.3. Since $i(V)$ is dense in H, then i^* is injective. Since i is injective and V is reflexive, $i^*(H') = H'$ is dense in V'. We can thus identify H' with a dense subspace of V'. By the Riesz representation theorem (B.4), we arrive at the identifications

$$V \subset H \equiv H' \subset V'. \tag{3.2}$$

In particular, we can identify the action of a linear functional in V' acting on an element of V with the scalar product of the two in H, whenever they both make sense. More precisely, if $\langle \cdot\,, \cdot \rangle_{V',V}$ denotes the action of a linear functional in V' on an element of V, and $(\cdot\,, \cdot)$ is the inner product on H inherited from $L^2(\Omega)$, then

$$\langle i^*(\boldsymbol{f})\,, \boldsymbol{u}\rangle_{V',V} = \langle \boldsymbol{f}\,, \boldsymbol{u}\rangle_{V',V} = (\boldsymbol{f}\,, \boldsymbol{u}), \quad \forall\, \boldsymbol{f} \in H,\ \forall\, \boldsymbol{u} \in V. \tag{3.3}$$

Remark 3.4. *[Brezis [2] §5.2.] Since V is also a Hilbert space, we could use the Riesz representation theorem* (B.4) *to identify V' with V. The point is that the dual space is defined to be the set of all bounded linear functionals, and we then choose how to characterize this space. For a Hilbert space X, one way to do this is with the Riesz representation theorem, where the isometry from X to X' is viewed as the identity map. However, it doesn't have to be done in this way. If we do choose to identify V with V' in this way, then obviously* (3.2) *is false. Consequently, we cannot identify H with H' and V with V' simultaneously. In this case, we choose to identify H with H'. Note that there is still an isometry from V to V', but in this case it is not viewed as the identity map.*

Now, if $\boldsymbol{u}$ is a weak solution, by considering those $\boldsymbol{\varphi}(x,t) = \phi(t)\boldsymbol{\psi}(x)$, we see that $\boldsymbol{u}$ must satisfy

$$- \int_0^T (\boldsymbol{u}(t)\,, \boldsymbol{\psi})\, \phi'(t)\, \mathrm{d}t \,+ \int_0^T (\nabla \boldsymbol{u}(t)\,, \nabla \boldsymbol{\psi})\, \phi(t)\, \mathrm{d}t + \int_0^T b(\boldsymbol{u}(t)\,, \boldsymbol{u}(t)\,, \boldsymbol{\psi})\, \phi(t)\, \mathrm{d}t = 0. \tag{3.4}$$

In fact, the converse is also true. It follows from the fact that any $\boldsymbol{\varphi} \in C^\infty_{0,0}(\Omega_T)$ can be approximated by a finite linear combination of functions of the form $\phi(t)\boldsymbol{\psi}(x)$. The proof, while not difficult, is lengthy. We omit it and refer the reader to Galdi [10] Lemma 2.3.

3.1 Proof of Theorem 3.1

Let us now prove Theorem 3.1. The idea, due to Hopf [13], is to apply the Galerkin method. We construct a sequence of approximate solutions on nested finite dimensional subspaces where we can apply known theory, and then attempt to pass to the limit, using energy estimates and compactness arguments to prove the required convergence to a full solution. We give our own version of the proof, using ideas from Temam [30], Ladyzhenskaya [17], Seregin [25] and Galdi [10].

Proof (of Theorem 3.1).
Step 1: (Construction of approximations).

Since H is a separable Hilbert space, and $C^\infty_{0,0}(\Omega)$ is dense in H, there exists[2] a set $\{\boldsymbol{\psi}_m\}_{m=1}^\infty \subset C^\infty_{0,0}(\Omega)$ that is dense in H and orthonormal with respect to the inner product on $L^2(\Omega)$. For each $m \geq 1$, define an approximate solution on the finite dimensional subspace spanned by $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_m\}$ by:

$$\boldsymbol{u}_m = \sum_{i=1}^m f_{i,m}(t)\, \boldsymbol{\psi}_i, \tag{3.5}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_m\,, \boldsymbol{\psi}_j) + (\nabla \boldsymbol{u}_m\,, \nabla \boldsymbol{\psi}_j) + b(\boldsymbol{u}_m\,, \boldsymbol{u}_m\,, \boldsymbol{\psi}_j) = 0 \tag{3.6}$$

for each $j = 1, \ldots, m$, and

$$\boldsymbol{u}_m(0) = \boldsymbol{u}_{0,m} = \sum_{i=1}^m (\boldsymbol{u}_0\,, \boldsymbol{\psi}_i). \tag{3.7}$$

Note that $\boldsymbol{u}_{0,m} \to \boldsymbol{u}_0$ in H.

Plugging (3.5) into (3.6) and using the orthonormality of $\boldsymbol{\psi}_i$, we obtain a system of ordinary (nonlinear) differential equations:

$$f'_{j,m}(t) \;+\; \sum_{i=1}^m (\nabla \boldsymbol{\psi}_i\,, \nabla \boldsymbol{\psi}_j)\, f_{i,m}(t) + \sum_{i,\,l=1}^m b(\boldsymbol{\psi}_i\,, \boldsymbol{\psi}_l\,, \boldsymbol{\psi}_j)\, f_{i,m}(t)\, f_{l,m}(t) = 0. \tag{3.8}$$

The initial condition is $f_{j,m}(0) = j$-th component of $\boldsymbol{u}_{0,m}$.

By Picard's existence theorem for ordinary differential equations[3], there exists a unique solution to the system defined on $[0, t_{\max})$. If $t_{\max} < T$, then $|f_{j,m}(t)| \to \infty$ as $t \to t_{\max}$, for some j. We claim that this is not possible.

Step 2: (Energy estimates).
Multiply (3.6) by $f_{j,m}(t)$ and sum over $j = 1, \ldots, m$, to obtain

$$(\boldsymbol{u}'_m\,, \boldsymbol{u}_m) + (\nabla \boldsymbol{u}_m\,, \nabla \boldsymbol{u}_m) + b(\boldsymbol{u}_m\,, \boldsymbol{u}_m\,, \boldsymbol{u}_m) = 0.$$

By Lemma 3.2,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m\|^2_{2,\Omega} + \|\nabla \boldsymbol{u}_m\|^2_{2,\Omega} = 0. \tag{3.9}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_m\|^2_{2,\Omega} \leq 0,$$

[2] Technically, this basis is only in H. However, it is possible to show that we can take this basis to be in $C^\infty_{0,0}(\Omega)$, and we thus assume this fact for simplicity. See Galdi [10] Lemma 2.3 for details.

[3] See, for example, Walter [32] §6.

which we may then integrate from 0 to t to obtain

$$\|\boldsymbol{u}_m(t)\|_{2,\Omega}^2 \leq \|\boldsymbol{u}_{0,m}\|_{2,\Omega}^2 \leq \|\boldsymbol{u}_0\|_{2,\Omega}^2 \tag{3.10}$$

$$\implies \sup_{t\in[0,T]} \|\boldsymbol{u}_m(t)\|_{2,\Omega} \leq \|\boldsymbol{u}_0\|_{2,\Omega}\,. \tag{3.11}$$

Thus, there exists a constant M such that

$$\sup_{t\in[0,T]} |f_{j,m}(t)| \leq M, \quad \forall\, j = 1, \ldots, m.$$

It follows that $t_{\max} = T$. Since (3.11) is true for any m, the sequence $\{\boldsymbol{u}_m\}_{m=1}^{\infty}$ is bounded in $L^\infty(0,T\,;H)$.

Integrate (3.9) from 0 to T to obtain

$$\|\boldsymbol{u}_m(T)\|_{2,\Omega}^2 - \|\boldsymbol{u}_{0,m}\|_{2,\Omega}^2 + 2\int_0^T \|\nabla \boldsymbol{u}_m(t)\|_{2,\Omega}^2 \,\mathrm{d}t = 0$$

$$\implies \int_0^T \|\nabla \boldsymbol{u}_m(t)\|_{2,\Omega}^2 \,\mathrm{d}t \;\leq\; \|\boldsymbol{u}_{0,m}\|_{2,\Omega}^2 \;\leq\; \|\boldsymbol{u}_0\|_{2,\Omega}^2\,, \tag{3.12}$$

and the sequence $\{\boldsymbol{u}_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T\,;V)$.

By the Banach-Alaoglu theorem (B.5), and by passing to subsequences if necessary, there exists $\boldsymbol{u} \in L^\infty(0,T\,;H) \cap L^2(0,T\,;V)$ such that $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$ weak-star in $L^\infty(0,T\,;H)$ and weakly in $L^2(0,T\,;V)$. That is,

$$\begin{cases} \boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \quad \text{in } L^2(0,T\,;V), \\ \\ \boldsymbol{u}_m \overset{\star}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^\infty(0,T\,;H). \end{cases} \tag{3.13}$$

Thus, condition 1 of Definition 2.4 holds.

The first condition in (3.13) is

$$\int_0^T \langle \boldsymbol{v}(t)\,, \boldsymbol{u}_m(t) - \boldsymbol{u}(t)\rangle_{V',V} \,\mathrm{d}t \to 0, \quad \forall\, \boldsymbol{v} \in L^2(0,T\,;V').$$

By the identification (3.3),

$$\int_0^T (\boldsymbol{u}_m(t) - \boldsymbol{u}(t)\,, \boldsymbol{v}(t)) \,\mathrm{d}t \to 0, \quad \forall\, \boldsymbol{v} \in L^2(0,T\,;H)$$

$$\implies \boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ in } L^2(0,T\,;H). \tag{3.14}$$

Step 3: (Passing to the limit).
We now want to show that $\boldsymbol{u}$ given by (3.13) satisfies condition 3 of Definition 2.4. To do this, we will need to pass to the limit in (3.6). However, the convergence proved so far will not be enough, in particular for the nonlinear term $b(\boldsymbol{u}_m\,,\boldsymbol{u}_m\,,\boldsymbol{\varphi}_j)$. In fact, we will need that $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in $L^2(0,T\,;H)$, which we will assume for now, and prove in Proposition 3.6.

Multiply (3.6) by $\phi(t) \in C_0^\infty((0,T))$, and integrate by parts:

$$-\int_0^T (\boldsymbol{u}_m(t)\,,\boldsymbol{\psi}_j)\,\phi'(t)\,\mathrm{d}t + \int_0^T (\nabla\boldsymbol{u}_m(t)\,,\nabla\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t + \int_0^T b(\boldsymbol{u}_m(t)\,,\boldsymbol{u}_m(t)\,,\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t = 0. \quad (3.15)$$

We can pass to the limit in the first term using the weak convergence (3.14). For the second term, we can put the spatial derivatives onto $\boldsymbol{\psi}_j$ using integration by parts, pass to the limit using (3.14), then put the spatial derivative back onto $\boldsymbol{u}$. For the nonlinear term, we use the Cauchy-Schwarz inequality, estimate (3.11), and the fact $\boldsymbol{u}_m \to \boldsymbol{u}$ in $L^2(0,T\,;H)$, proved in Proposition 3.6. Explicitly,

$$\begin{aligned}
&\int_0^T \int_\Omega |(u_m)_i(u_m)_k - u_iu_k| \left|\frac{\partial(\psi_j)_i}{\partial x_k}\phi\right| \mathrm{d}x\,\mathrm{d}t \\
&\le \|\phi\cdot\nabla\boldsymbol{\psi}_j\|_{\infty,\Omega_T}\int_0^T\int_\Omega |(u_m)_i(u_m)_k - (u_m)_iu_k| + |(u_m)_iu_k - u_iu_k|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\,\|\boldsymbol{u}_0\|_{2,\Omega}\cdot\|\boldsymbol{u}_m - \boldsymbol{u}\|_{L^2(0,T;H)} \to 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^T b(\boldsymbol{u}_m(t)\,,\boldsymbol{u}_m(t)\,,\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t = -\int_0^T b(\boldsymbol{u}_m(t)\,,\boldsymbol{\psi}_j\,,\boldsymbol{u}_m(t))\,\phi(t)\,\mathrm{d}t \\
&= -\sum_{i,k=1}^3\int_0^T\int_\Omega (u_m)_i\frac{\partial(\psi_j)_i}{\partial x_k}(u_m)_k\,\phi(t)\,\mathrm{d}x\,\mathrm{d}t \to -\sum_{i,k=1}^3\int_0^T\int_\Omega u_i\frac{\partial(\psi_j)_i}{\partial x_k}u_k\,\phi(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T b(\boldsymbol{u}(t)\,,\boldsymbol{u}(t)\,,\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t.
\end{aligned}$$

Now, we pass to the limit in (3.15) to obtain

$$
-\int_0^T (\boldsymbol{u}(t)\,,\boldsymbol{\psi}_j)\,\phi'(t)\,\mathrm{d}t + \int_0^T (\nabla\boldsymbol{u}(t)\,,\nabla\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t + \int_0^T b(\boldsymbol{u}(t)\,,\boldsymbol{u}(t)\,,\boldsymbol{\psi}_j)\,\phi(t)\,\mathrm{d}t = 0. \quad (3.16)
$$

By linearity, this is valid for each finite linear combination of the $\boldsymbol{\psi}_j$. Then, by continuity, it is valid for every $\boldsymbol{\psi} \in V$ and (3.4) holds, and thus so does condition 3 of Definition 2.4.

Now, Proposition 3.6 will also imply that $\boldsymbol{u}_m(\cdot, t) \rightharpoonup \boldsymbol{u}(\cdot, t)$ in $L^2(\Omega)$ uniformly in $t \in [0, T]$. In other words, $\boldsymbol{u}_m(\cdot, t)$ converges weakly to $\boldsymbol{u}(\cdot, t)$ in $L^2(\Omega)$ for each $t \in [0, T]$, and this convergence is independent of the specific value of t. In particular, we will show in Proposition 3.6 that condition 2 of Definition 2.4 holds.

It remains to verify that the weak energy inequality holds. Integrating (3.9), we find that

$$
\frac{1}{2}\|\boldsymbol{u}_m(t)\|_{2,\Omega}^2 + \int_0^t \|\nabla\boldsymbol{u}_m(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau = \frac{1}{2}\|\boldsymbol{u}_{0,m}\|_{2,\Omega}^2. \quad (3.17)
$$

On the one hand, $\|\boldsymbol{u}_{0,m}\|_{2,\Omega} \to \|\boldsymbol{u}_0\|_{2,\Omega}$. On the other hand, since $\boldsymbol{u}_m(\cdot, t) \rightharpoonup \boldsymbol{u}(\cdot, t)$ in $L^2(\Omega)$ uniformly in $t \in [0, T]$ (proved in Proposition 3.6), we know that (cf. Remark 2.3)

$$
\|\boldsymbol{u}(t)\|_{2,\Omega}^2 \leq \liminf_{m\to\infty} \|\boldsymbol{u}_m(t)\|_{2,\Omega}^2\,, \quad \forall\, t \in [0, T].
$$

Similarly, by (3.13),

$$
\int_0^T \|\nabla\boldsymbol{u}(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau \leq \liminf_{m\to\infty} \int_0^T \|\nabla\boldsymbol{u}_m(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau.
$$

Thus, taking the $\liminf_{m\to\infty}$ of (3.17), we obtain

$$
\frac{1}{2}\|\boldsymbol{u}(t)\|_{2,\Omega}^2 + \int_0^t \|\nabla\boldsymbol{u}(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau \leq \frac{1}{2}\|\boldsymbol{u}_0\|_{2,\Omega}^2, \quad \forall t \in [0, T]. \quad (3.18)
$$

We thus conclude that the weak energy inequality holds for all $t \in [0, T]$, and condition 5 of Definition 2.4 holds. By Remark 2.3, condition 4 of Definition 2.4 also holds. Theorem 3.1 is proved. □

It remains to prove that $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in $L^2(0, T\,; H)$. To show this, we will need the following Friedrichs-type inequality.

Lemma 3.5. *[Galdi [9] Lemma II.4.2.]*
Let $\boldsymbol{v} \in V$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and functions $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N \in H$ such that

$$\|\boldsymbol{v}\|_{2,\Omega}^2 \leq \sum_{j=1}^{N} |(\boldsymbol{v}\,,\boldsymbol{\omega}_j)|^2 \,+\, \varepsilon\,\|\nabla \boldsymbol{v}\|_{2,\Omega}^2.$$

PROOF. By density arguments, it is enough to prove the result for $\boldsymbol{v} \in C_{0,0}^{\infty}(\Omega)$. By extending $\boldsymbol{v}$ to be zero outside of Ω, and considering a cube that contains $\overline{\Omega}$, it is enough to prove the statement when the domain is a cube Q with side length L.

Suppose first that Q_i is a cube with side length a. By translation and rotation, we can assume one of the corners is located at the origin and the edges lie along the positive Cartesian axes. For any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in Q_i$,

$$\boldsymbol{v}(x)-\boldsymbol{v}(y) = \int_{y_1}^{x_1} \frac{\partial}{\partial \xi}\boldsymbol{v}(\xi, x_2, x_3)\,\mathrm{d}\xi + \int_{y_2}^{x_2} \frac{\partial}{\partial \eta}\boldsymbol{v}(y_1, \eta, x_3)\,\mathrm{d}\eta + \int_{y_3}^{x_3} \frac{\partial}{\partial \zeta}\boldsymbol{v}(y_1, y_2, \zeta)\,\mathrm{d}\zeta.$$

Take the $\mathbb{R}^3$ inner product of this equation with itself, use the Cauchy-Schwarz inequality and apply the inequality[4]

$$(\alpha_1+\alpha_2+\alpha_3)^2 \leq 3\,(\alpha_1^2+\alpha_2^2+\alpha_3^2)$$

to find that

$$\begin{aligned}|\boldsymbol{v}(x)|^2-2|\boldsymbol{v}(x)||\boldsymbol{v}(y)|+|\boldsymbol{v}(y)|^2 \;\leq\; & 3\left|\int_{y_1}^{x_1} \frac{\partial}{\partial \xi}\boldsymbol{v}(\xi, x_2, x_3)\,\mathrm{d}\xi\right|^2 + 3\left|\int_{y_2}^{x_2} \frac{\partial}{\partial \eta}\boldsymbol{v}(y_1, \eta, x_3)\,\mathrm{d}\eta\right|^2 \\ & +\; 3\left|\int_{y_3}^{x_3} \frac{\partial}{\partial \zeta}\boldsymbol{v}(y_1, y_2, \zeta)\,\mathrm{d}\zeta\right|^2.\end{aligned}$$

By the Cauchy-Schwarz inequality and the fact that $|x_i - y_i| \leq a$, we can bound the right-hand side by

$$3a\left[\int_0^a \left|\frac{\partial}{\partial \xi}\boldsymbol{v}(\xi, x_2, x_3)\right|^2 \mathrm{d}\xi + \int_0^a \left|\frac{\partial}{\partial \eta}\boldsymbol{v}(y_1, \eta, x_3)\right|^2 \mathrm{d}\eta + \int_0^a \left|\frac{\partial}{\partial \zeta}\boldsymbol{v}(y_1, y_2, \zeta)\right|^2 \mathrm{d}\zeta\right].$$

Integrating with respect to x and y over Q_i, we see that

$$2a^3\|\boldsymbol{v}\|_{2,Q_i}^2 - 2\left(\int_{Q_i} |\boldsymbol{v}(x)|\,\mathrm{d}x\right)^2 \leq 3a^5\|\nabla \boldsymbol{v}\|_{2,Q_i}^2.$$

We thus obtain the Poincaré-type inequality

$$\|\boldsymbol{v}\|_{2,Q_i}^2 \leq \frac{1}{a^3}\left(\int_{Q_i} |\boldsymbol{v}(x)|\,\mathrm{d}x\right)^2 + \frac{3}{2}a^2\|\nabla \boldsymbol{v}\|_{2,Q_i}^2. \tag{3.19}$$

[4]This is just the Cauchy-Schwarz inequality in $\mathbb{R}^3$.

Now subdivide the original cube Q into $N = K^3$ identical smaller cubes Q_i each with side length $a = L/K$. By (3.19),

$$\|\boldsymbol{v}\|_{2,Q_i}^2 \leq \frac{K^3}{L^3}\left(\int_{Q_i} |\boldsymbol{v}(x)| \,\mathrm{d}x\right)^2 + \frac{3L^2}{2K^2}\|\nabla \boldsymbol{v}\|_{2,Q_i}^2.$$

Summing over $i = 1, \ldots, N$, we obtain

$$\|\boldsymbol{v}\|_{2,Q}^2 \leq \sum_{i=1}^{N} \frac{K^3}{L^3}\left(\int_{Q_i} |\boldsymbol{v}(x)| \,\mathrm{d}x\right)^2 + \frac{3L^2}{2K^2}\|\nabla \boldsymbol{v}\|_{2,Q}^2\,.$$

Choose

$$\boldsymbol{w}_i(x) = \left(\frac{K^3}{L^3}\right)^{\frac{1}{2}} \mathbb{1}_{Q_i}(x) \in L^2(Q),$$

where $\mathbb{1}_{Q_i}$ is the indicator function on Q_i. By Theorem 2.1, we can decompose $\boldsymbol{w}_i$ as $\boldsymbol{w}_i = \boldsymbol{\omega}_i + \nabla p_i$, with $\boldsymbol{\omega}_i \in H$. Since $\boldsymbol{v} \in V \subset H$, then $(\boldsymbol{v}\,, \boldsymbol{w}_i) = (\boldsymbol{v}\,, \boldsymbol{\omega}_i)$. The result follows by choosing $N = K^3$ so large such that $K \geq L\sqrt{3/2\varepsilon}$. □

Proposition 3.6. *[Ladyzhenskaya [17] Theorem VI.13.]*
The sequence $\boldsymbol{u}_m$ defined in (3.13) *has a subsequence, also denoted by $\boldsymbol{u}_m$, that satisfies $\boldsymbol{u}_m \to \boldsymbol{u}$ strongly in $L^2(0, T\,; H)$.*

Proof. We know that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $L^2(0, T\,; H)$. By (3.10), we have that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $L^2(\Omega)$ for each fixed $t \in [0, T]$. Fix $i \geq 1$. By (3.11), the coefficients $f_{i,m}(t)$ are uniformly bounded on $[0, T]$. Integrate (3.6) from t to $t + \delta$ and repeatedly apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|f_{i,m}(t+\delta) - f_{i,m}(t)| &\equiv \left|\int_t^{t+\delta} \frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{u}_m(\tau)\,, \boldsymbol{\psi}_i) \,\mathrm{d}\tau\right| \\
&\leq \int_t^{t+\delta} \|\nabla \boldsymbol{u}_m(\tau)\|_{2,\Omega} \cdot \|\nabla \boldsymbol{\psi}_i\|_{2,\Omega} \,\mathrm{d}\tau \\
&\qquad + \|\boldsymbol{\psi}_i\|_{\infty,\Omega} \int_t^{t+\delta} \|\boldsymbol{u}_m(\tau)\|_{2,\Omega} \cdot \|\nabla \boldsymbol{u}_m(\tau)\|_{2,\Omega} \,\mathrm{d}\tau \\
&\leq C_i\sqrt{\delta}\left(\int_t^{t+\delta} \|\nabla \boldsymbol{u}_m(\tau)\|_{2,\Omega}^2 \,\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\qquad + C_i\sqrt{\delta}\,\|\boldsymbol{u}_m\|_{L^\infty(0,T;H)}\left(\int_t^{t+\delta} \|\nabla \boldsymbol{u}_m(\tau)\|_{2,\Omega}^2 \,\mathrm{d}\tau\right)^{\frac{1}{2}},
\end{aligned}$$

where $C_i = \max\{\|\boldsymbol{\psi}_i\|_{\infty,\Omega}\,, \|\nabla \boldsymbol{\psi}_i\|_{2,\Omega}\}$.

By (3.11) and (3.12), the right-hand side converges to zero as $\delta \to 0$, independent of m. Hence, for fixed $i \geq 1$, $f_{i,m}(t)$ forms a sequence of uniformly bounded and uniformly equicontinuous functions.

By the Arzelà-Ascoli theorem (B.6), for each i there exists a uniformly convergent subsequence, also denoted by $f_{i,m}(t)$. This sequence depends on i. However, by using the standard diagonalization argument[5], we extract a subsequence such that $f_{i,m}(t)$ converges uniformly to $f_i(t)$, for every $i \geq 1$. This then implies

$$(\boldsymbol{u}_m\,, \boldsymbol{\psi}) \to (\boldsymbol{u}\,, \boldsymbol{\psi}) \tag{3.20}$$

uniformly for $t \in [0,T]$, for any $\boldsymbol{\psi} \in \operatorname{span}\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k\}$. Since $\{\boldsymbol{\psi}_j\}_{j=1}^{\infty}$ is dense in H, then $\boldsymbol{u}_m$ converges weakly in H, uniformly for $t \in [0,T]$. To see this, take any $\boldsymbol{v} \in H$, approximate by $\boldsymbol{\psi} \in \operatorname{span}\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k\}$, apply Cauchy-Schwarz, use (3.11) and pass to the limit.

In particular, condition 2 of Definition 2.4 holds. Indeed, for $\boldsymbol{w} \in L^2(\Omega)$, consider the continuous functions

$$S_{w,m}(t) = \int_\Omega \boldsymbol{u}_m(x,t) \cdot \boldsymbol{w}(x)\,\mathrm{d}x.$$

By Theorem 2.1, $\boldsymbol{w} = \boldsymbol{\psi} + \nabla q$, with $\boldsymbol{\psi} \in H$ and $q \in L^2(\Omega)$. Since $\boldsymbol{u}_m \in H$, and H is orthogonal to $G(\Omega)$, then $(\boldsymbol{u}_m\,, \nabla q) = 0$ for every m. By the weak convergence in H,

$$\begin{aligned} S_{w,m}(t) = \int_\Omega \boldsymbol{u}_m(x,t) \cdot \boldsymbol{\psi}(x)\,\mathrm{d}x \to &\int_\Omega \boldsymbol{u}(x,t) \cdot \boldsymbol{\psi}(x)\,\mathrm{d}x \\ &= \int_\Omega \boldsymbol{u}(x,t) \cdot \boldsymbol{w}(x)\,\mathrm{d}x = S_w(t), \end{aligned}$$

and the uniformity in time of this convergence ensures that S_w is continuous. By the same argument, $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$ weakly in $L^2(\Omega)$, uniformly in $t \in [0,T]$.

By Lemma 3.5,

$$\begin{aligned} \int_0^T \|\boldsymbol{u}_n(\tau) - \boldsymbol{u}_m(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau \;\leq\; &\sum_{j=1}^{N} \int_0^T (\boldsymbol{u}_n(\tau) - \boldsymbol{u}_m(\tau)\,, \boldsymbol{\omega}_j)^2\,\mathrm{d}\tau \\ &+ \varepsilon \int_0^T \|\nabla\boldsymbol{u}_n(\tau) - \nabla\boldsymbol{u}_m(\tau)\|_{2,\Omega}^2\,\mathrm{d}\tau. \end{aligned} \tag{3.21}$$

By estimate (3.12) and the uniformity in t of the weak convergence, the right-hand side of (3.21) is arbitrarily small for sufficiently large m and n. Thus, $\boldsymbol{u}_m$ is Cauchy in $L^2(0,T\,;H)$, and so converges strongly to $\boldsymbol{u}$ in $L^2(0,T\,;H)$. □

[5] See, for example, Bartle [1] §17.16.

Remark 3.7. *We have omitted details about the uniqueness of limits of the sequence* $\boldsymbol{u}_m$ *under different forms of convergence. However, it can be shown that* $\boldsymbol{u}$ *is the unique limit up to an arbitrary null set in* Ω_T*, see Heywood [12] and Temam [30] III.1.*

3.2 Comments

There are many different ways to prove Theorem 3.1, for example with fixed-point methods (Seregin [25] Theorem 3.5) or by semi-discretization in time (Temam [30] Theorem III.4.1). However, the basic idea of using the energy estimates and a compactness argument remains the same. Note that the solution $\boldsymbol{u}$ we obtained is unique in the sense that there can only be one solution obtained using the Galerkin method. This, however, says nothing about the uniqueness of solutions in general.

The proof is independent of the dimension of the space, except when using the Sobolev embedding theorem to prove continuity of the operator $b(\boldsymbol{u}\,,\boldsymbol{v}\,,\boldsymbol{w})$. In particular, this proof holds in dimension two. Furthermore, clearly it is the nonlinear term that is the main difficulty in this proof, and neglecting it means that we do not need Proposition 3.6 to prove the existence of Leray-Hopf weak solutions to the corresponding linear problem. It is also clear that the addition of an external force $\boldsymbol{F} \in L^2(0,T\,;V')$ in the equation of motion also does not cause any problems with the formulation or the proof.

Finally, it can be shown that there exists a function $P \in C(0,T\,;L^2(\Omega))$ such that

$$\boldsymbol{u}(t) - \boldsymbol{u}_0 + \int_0^t (\boldsymbol{u}(\tau)\cdot\nabla)\boldsymbol{u}(\tau)\,\mathrm{d}\tau - \Delta\left(\int_0^t \boldsymbol{u}(\tau)\,\mathrm{d}\tau\right) + \nabla P(t) = 0$$

for each $t \in [0,T]$. Differentiating this expression in the sense of distributions and setting $p = \partial_t P$, we obtain a distribution p such that

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = 0$$

is satisfied in the sense of distributions. The details of this can be found in Temam [30] Ch.III §3.

4 Uniqueness and Regularity

Throughout this chapter, we use the letter C to denote any constant that can be found using quantities already known, so that its value may change from line to line.

4.1 Three-dimensional Case

Having proved the existence of Leray-Hopf weak solutions in Chapter 3, we now consider the question of their uniqueness. We first start with the simplest case that the velocity $\boldsymbol{u}$ and pressure p are smooth. Suppose $(\boldsymbol{v}, q)$ is another smooth solution pair. Let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and $r = p - q$. Then

$$\partial_t \boldsymbol{w} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} = -\nabla r + \Delta \boldsymbol{w}.$$

Taking the $L^2(\Omega)$ inner product with $\boldsymbol{w}$, we obtain

$$(\boldsymbol{w}\,, \partial_t \boldsymbol{w}) + b(\boldsymbol{u}\,, \boldsymbol{w}\,, \boldsymbol{w}) + b(\boldsymbol{w}\,, \boldsymbol{v}\,, \boldsymbol{w}) = -(\boldsymbol{w}\,, \nabla r) + (\boldsymbol{w}\,, \Delta \boldsymbol{w})$$

$$\implies (\boldsymbol{w}\,, \partial_t \boldsymbol{w}) + b(\boldsymbol{u}\,, \boldsymbol{w}\,, \boldsymbol{w}) + b(\boldsymbol{w}\,, \boldsymbol{v}\,, \boldsymbol{w}) = -(\boldsymbol{w}\,, \nabla r) - (\nabla \boldsymbol{w}\,, \nabla \boldsymbol{w}). \tag{4.1}$$

The second and fourth terms vanish by Lemma 3.2 and the fact that $\operatorname{div} \boldsymbol{w} = 0$. Thus,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{2,\Omega}^2 + \|\nabla \boldsymbol{w}\|_{2,\Omega}^2 = |b(\boldsymbol{w}\,, \boldsymbol{v}\,, \boldsymbol{w})| \tag{4.2}$$

$$\implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{2,\Omega}^2 \le C\,\|\nabla \boldsymbol{v}\|_{\infty,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega}^2\,. \tag{4.3}$$

Since $\boldsymbol{v}$ is smooth, $\|\nabla \boldsymbol{v}\|_{\infty,\Omega} < \infty$, and we can apply Gronwall's inequality (B.1) to find that

$$\|\boldsymbol{w}(t)\|_{2,\Omega}^2 \le \|\boldsymbol{w}_0\|_{2,\Omega}^2\, e^{Ct}, \quad \forall\, t \ge 0.$$

Since $\boldsymbol{w}_0 \equiv 0$, it follows that $\boldsymbol{w}(x,t) = 0$ in Ω_T. Thus, if a smooth solution exists, it is unique.

Note that we did not use the full smoothness of $\boldsymbol{w}$ to deduce its uniqueness. Indeed, the two places where we have used regularity properties not already attained by Leray-Hopf weak solutions are (4.1) in writing down $(\boldsymbol{w}\,, \partial_t \boldsymbol{w})$, and (4.3) in estimating the nonlinear term with $\|\nabla \boldsymbol{v}\|_{\infty,\Omega}$. The latter can be dealt with by estimating $|b(\boldsymbol{w}\,, \boldsymbol{v}\,, \boldsymbol{w})|$ in a different way. Following Seregin [25] Proposition 7.15, the Cauchy-Schwarz inequality implies

$$|b(\boldsymbol{w}\,, \boldsymbol{v}\,, \boldsymbol{w})| \le \|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}^2\|_{2,\Omega} = \|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{4,\Omega}^2\,. \tag{4.4}$$

Lemma 4.1 (Ladyzhenskaya's inequality)**.** *Let* $\boldsymbol{u} \in W_0^{1,2}(\Omega)$*. Then there exists a* *constant C independent of $\boldsymbol{u}$ such that*

$$\|\boldsymbol{u}\|_{4,\Omega} \leq C\,\|\boldsymbol{u}\|_{2,\Omega}^{\frac{1}{4}} \cdot \|\nabla \boldsymbol{u}\|_{2,\Omega}^{\frac{3}{4}} \tag{4.5}$$

Proof. By Sobolev embedding (B.10) and Poincaré's inequality (B.9),

$$\|\boldsymbol{u}\|_{6,\Omega} \leq C\,\|\nabla \boldsymbol{u}\|_{2,\Omega}. \tag{4.6}$$

In addition, we have an interpolation inequality in $L^r(\Omega)$: if $1 \leq q < r < s \leq \infty$, then there exists $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}$ and

$$\|\boldsymbol{u}\|_{r,\Omega} \leq \|\boldsymbol{u}\|_{q,\Omega}^{\theta} \cdot \|\boldsymbol{u}\|_{s,\Omega}^{1-\theta}. \tag{4.7}$$

To prove this, apply Hölder's inequality (2.1) to

$$f = |\boldsymbol{u}|^{r\theta}, \quad g = |\boldsymbol{u}|^{r(1-\theta)}, \quad p = \frac{q}{r\theta}, \quad p' = \frac{s}{r(1-\theta)}\,.$$

Now, with $r = 4$, $q = 2$, $s = 6$ and $\theta = 1/4$ in (4.7), we find that

$$\|\boldsymbol{u}\|_{4,\Omega} \leq \|\boldsymbol{u}\|_{2,\Omega}^{\frac{1}{4}} \cdot \|\boldsymbol{u}\|_{6,\Omega}^{\frac{3}{4}}.$$

From this and (4.6), the result follows. □

By Lemma 4.1, we can estimate the right-hand side of (4.4) by

$$|b(\boldsymbol{w}\,,\boldsymbol{v}\,,\boldsymbol{w})| \leq C\,\|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega}^{\frac{1}{2}} \cdot \|\nabla \boldsymbol{w}\|_{2,\Omega}^{\frac{3}{2}}. \tag{4.8}$$

Apply Young's inequality (B.8) with

$$a = C_1\,\|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega}^{\frac{1}{2}}\,, \quad b = C_2\,\|\nabla \boldsymbol{w}\|_{2,\Omega}^{\frac{3}{2}}\,, \quad p = 4\,,\, p' = 4/3\,, \quad C_1C_2 = C.$$

Then

$$\begin{aligned}|b(\boldsymbol{w}\,,\boldsymbol{v}\,,\boldsymbol{w})| &\leq \frac{1}{4}\left(C_1\,\|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega}^{\frac{1}{2}}\right)^4 + \frac{3}{4}\left(C_2\,\|\nabla \boldsymbol{w}\|_{2,\Omega}^{\frac{3}{2}}\right)^{\frac{4}{3}} \\ &= C_3\,\|\nabla \boldsymbol{v}\|_{2,\Omega}^4 \cdot \|\boldsymbol{w}\|_{2,\Omega}^2 + C_4\,\|\nabla \boldsymbol{w}\|_{2,\Omega}^2\,.\end{aligned}$$

Choose C_1 and C_2 such that $C_4 = 1$ and let $C(t) = C_3\,\|\nabla \boldsymbol{v}(t)\|_{2,\Omega}^4$.

We now make the following assumption:

$$\nabla \boldsymbol{v} \in L^\infty(0,T\,; L^2(\Omega)). \tag{4.9}$$

Then there exists C such that $C(t) \leq C$, and hence

$$|b(\boldsymbol{w}\,,\boldsymbol{v}\,,\boldsymbol{w})| \leq C\,\|\boldsymbol{w}\|_{2,\Omega}^2 + \|\nabla \boldsymbol{w}\|_{2,\Omega}^2\,. \tag{4.10}$$

We now show that the term $(\boldsymbol{w}\,,\partial_t\boldsymbol{w})$ also makes sense.

Lemma 4.2. *Suppose that a Leray-Hopf weak solution $\boldsymbol{w}$ satisfies*

$$\nabla\boldsymbol{w} \in L^\infty(0,T\,;L^2(\Omega)).$$

Then

$$\partial_t\boldsymbol{w} \in L^2(0,T\,;V'). \tag{4.11}$$

Proof. By definition,

$$\partial_t\boldsymbol{w} \in L^2(0,T\,;V') \iff \int_0^T \|\partial_t\boldsymbol{w}\|_{V'}^2\,\mathrm{d}t < \infty$$

$$\iff \int_0^T \Big[\sup\{\,|(\partial_t\boldsymbol{w}\,,\boldsymbol{\psi})| \mid \boldsymbol{\psi}\in V\,,\|\nabla\boldsymbol{\psi}\|_{2,\Omega}=1\}\Big]^2\,\mathrm{d}t < \infty.$$

Using the equation of motion (3.4), this is true if

$$\int_0^T \sup\Big\{\,|b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})|^2 \mid \boldsymbol{\psi}\in V\,,\|\nabla\boldsymbol{\psi}\|_{2,\Omega}=1\Big\}\;\mathrm{d}t < \infty, \tag{4.12}$$

and

$$\int_0^T \sup\Big\{\,|(\nabla\boldsymbol{w}\,,\nabla\boldsymbol{\psi})|^2 \mid \boldsymbol{\psi}\in V\,,\|\nabla\boldsymbol{\psi}\|_{2,\Omega}=1\Big\}\;\mathrm{d}t < \infty. \tag{4.13}$$

Condition (4.13) is true by the Cauchy-Schwarz inequality and the assumption $\nabla\boldsymbol{w} \in L^\infty(0,T\,;L^2(\Omega))$. In fact, (4.13) is true for any Leray-Hopf weak solution, since we only require $\nabla\boldsymbol{w} \in L^2(0,T\,;L^2(\Omega))$, and any Leray-Hopf weak solution satisfies $\boldsymbol{w}\in L^2(0,T\,;V) \implies \nabla\boldsymbol{w}\in L^2(0,T\,;L^2(\Omega))$.

To see that the condition (4.12) also holds, we use the generalized Hölder inequality (2.2), Sobolev embedding (B.10) and Poincaré's inequality (B.9) to estimate

$$\begin{aligned}|b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})| = |b(\boldsymbol{w}\,,\boldsymbol{\psi}\,,\boldsymbol{w})| &\le C\,\|\boldsymbol{w}\|_{4,\Omega}^2\cdot\|\nabla\boldsymbol{\psi}\|_{2,\Omega}\\ &\le C\,\|\nabla\boldsymbol{w}\|_{2,\Omega}^2\,.\end{aligned} \tag{4.14}$$

For condition (4.12) to hold, we need $\nabla\boldsymbol{w}\in L^4(0,T\,;L^2(\Omega))$. By assumption, we know $\nabla\boldsymbol{w}\in L^\infty(0,T\,;L^2(\Omega)) \subset L^4(0,T\,;L^2(\Omega))$, and the result follows. □

Using Lemma 4.2 and (3.4), we know that

$$\int_0^T \Big[(\partial_t\boldsymbol{w}\,,\boldsymbol{\psi}) + (\nabla\boldsymbol{w}\,,\nabla\boldsymbol{\psi}) + b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})\Big]\phi(t)\,\mathrm{d}t = 0 \tag{4.15}$$

for every $\boldsymbol{\psi}\in C_{0,0}^\infty(\Omega)$ and $\phi\in C_0^\infty((0,T))$. By the fundamental lemma of calculus of variations (B.7) and continuity arguments, it follows that

$$(\partial_t \boldsymbol{w}\,,\boldsymbol{\psi}) + (\nabla \boldsymbol{w}\,,\nabla \boldsymbol{\psi}) + b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi}) = 0, \quad \forall\, \boldsymbol{\psi} \in V.$$

Now, we employ the same argument used to prove the uniqueness of smooth solutions. Using (4.2) and (4.10), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{2,\Omega}^2 + \|\nabla \boldsymbol{w}\|_{2,\Omega}^2 \leq C\,\|\boldsymbol{w}\|_{2,\Omega}^2 + \|\nabla \boldsymbol{w}\|_{2,\Omega}^2$$

$$\implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{2,\Omega}^2 \leq C\,\|\boldsymbol{w}\|_{2,\Omega}^2\,,$$

and we can apply Gronwall's inequality (B.1) to conlcude that $\boldsymbol{w} \equiv 0$.

This result motivates the following definition:

Definition 4.3. A Leray-Hopf weak solution $\boldsymbol{u}$ is called a *strong solution* if it satisfies

$$\nabla \boldsymbol{u} \in L^\infty(0,T\,;L^2(\Omega)). \tag{4.16}$$

We have thus proved the following proposition:

Proposition 4.4. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two strong solutions to the initial-boundary value problem, with initial data $\boldsymbol{u}_0 \in V$. Then $\boldsymbol{u} \equiv \boldsymbol{v}$.*

However, we can do better. Equation (4.2) suggests that it suffices for just $\boldsymbol{v}$ to be a strong solution, with $\boldsymbol{u}$ a Leray-Hopf weak solution. Indeed, the nonlinear term can then be estimated in exactly the same fashion. It is not immediately clear that the term $(\boldsymbol{w}\,,\partial_t \boldsymbol{w})$ makes sense since we do not know whether $\boldsymbol{u}$ has enough regularity in time. Nonetheless, the result holds, though the proof is long and technical. We assume the result here and refer the reader to Seregin [25] Theorem 7.14 for the details.

Theorem 4.5 (Weak-Strong Uniqueness)**.**
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be Leray-Hopf weak solutions to the initial-boundary value problem with initial data $\boldsymbol{u}_0 \in V$. If $\boldsymbol{v}$ is a strong solution, then $\boldsymbol{u} \equiv \boldsymbol{v}$.

There are many other similar results regarding uniqueness. For example, see Temam [30] Theorem III.3.4, or Giga [11] Lemma 5.2 for the Ladyzhenskaya-Prodi-Serrin condition, which guarantees uniqueness in any dimension.

4.2 Linear and Two-dimensional Case

We now briefly discuss the question of uniqueness for two different cases. First, suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are Leray-Hopf weak solutions to the linear problem in three dimensions and as usual let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. If we drop the nonlinear term from the equation of motion, Lemma 4.2 implies that $\partial_t \boldsymbol{w} \in L^2(0,T\,;V')$, and the proof of uniqueness goes through as before.

Suppose instead that $\boldsymbol{u}$ and $\boldsymbol{v}$ are Leray-Hopf weak solutions to the full problem in two dimensions. The two dimensional version of Ladyzhenskaya's inequality reads

$$\|\boldsymbol{u}\|_{4,\Omega}^2 \leq C\,\|\boldsymbol{u}\|_{2,\Omega} \cdot \|\nabla \boldsymbol{u}\|_{2,\Omega}\,. \tag{4.17}$$

Thus, instead of (4.8), we obtain

$$|b(\boldsymbol{w}\,,\boldsymbol{v}\,,\boldsymbol{w})| \leq C\,\|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega} \cdot \|\nabla \boldsymbol{w}\|_{2,\Omega}\,. \tag{4.18}$$

Apply Young's inequality (B.8) with

$$a = C_1\,\|\nabla \boldsymbol{v}\|_{2,\Omega} \cdot \|\boldsymbol{w}\|_{2,\Omega}\,, \quad b = C_2\,\|\nabla \boldsymbol{w}\|_{2,\Omega}\,, \quad p = p' = 2\,, \quad C_1 C_2 = C$$

$$\implies |b(\boldsymbol{w}\,,\boldsymbol{v}\,,\boldsymbol{w})| \leq C_3\,\|\nabla \boldsymbol{v}\|_{2,\Omega}^2 \cdot \|\boldsymbol{w}\|_{2,\Omega} + C_4\,\|\nabla \boldsymbol{w}\|_{2,\Omega}^2\,.$$

Choose C_1 and C_2 such that $C_4 = 1$, and let $C(t) = C_3\,\|\nabla \boldsymbol{v}\|_{2,\Omega}^2$. Then $C(t)$ is integrable, since $\boldsymbol{v} \in L^2(0,T\,;V) \implies \nabla \boldsymbol{v} \in L^2(0,T\,;L^2(\Omega))$.

To show that $\partial_t \boldsymbol{w} \in L^2(0,T\,;V')$, we proceed as in the proof of Lemma 4.2. Using (4.18), we see that (4.12) holds, since

$$\begin{aligned}\int_0^T \sup\left\{\,|b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})|^2 \mid \boldsymbol{\psi} \in V\,,\|\nabla \boldsymbol{\psi}\|_{2,\Omega} = 1\,\right\}\,\mathrm{d}t &\leq \int_0^T C\,\|\boldsymbol{w}\|_{2,\Omega}^2 \cdot \|\nabla \boldsymbol{w}\|_{2,\Omega}^2\,\mathrm{d}t \\ &\leq C\,\|\boldsymbol{w}\|_{L^\infty(0,T;H)}^2 \cdot \|\boldsymbol{w}\|_{L^2(0,T;V)}^2\,.\end{aligned}$$

The proof of uniqueness now carries though as usual.

Thus, the questions of existence and uniqueness of Leray-Hopf weak solutions for both the linear problem and the full two-dimensional problem are answered in a satisfactory manner.

4.3 Regularity

We now return to the three-dimensional case. Theorem 4.5 gave a sufficient condition for the uniqueness of Leray-Hopf weak solutions in the class of weak solutions. We now examine the regularity of solutions to see if they do satisfy this condition.

Proposition 4.6. *[Temam [30] Theorem III.3.3]*
Let $\boldsymbol{w}$ be a Leray-Hopf weak solution with initial data $\boldsymbol{w}_0 \in V$. Then

$$\boldsymbol{w} \in L^{8/3}(0,T\,;L^4(\Omega)) \quad \textit{and} \quad \partial_t \boldsymbol{w} \in L^{4/3}(0,T\,;V'). \tag{4.19}$$

Proof. By Ladyzhenskaya's inequality (4.5), we know that

$$\|\boldsymbol{w}(t)\|_{4,\Omega} \le C\,\|\boldsymbol{w}(t)\|_{2,\Omega}^{\frac{1}{4}} \cdot \|\nabla \boldsymbol{w}(t)\|_{2,\Omega}^{\frac{3}{4}}\,, \quad \text{a.e in } t \in [0,T]. \tag{4.20}$$

Since

$$\int_0^T \|\boldsymbol{w}(t)\|_{2,\Omega}^{\frac{2}{3}} \cdot \|\nabla \boldsymbol{w}(t)\|_{2,\Omega}^2 \,\mathrm{d}t \le \|\boldsymbol{w}\|_{L^\infty(0,T;H)}^{\frac{2}{3}} \cdot \|\boldsymbol{w}\|_{L^2(0,T;V)}^2 < \infty,$$

the right-hand side of (4.20) belongs to $L^{8/3}(0,T)$, and thus so does the left-hand side. Furthermore,

$$\partial_t \boldsymbol{w} \in L^{4/3}(0,T\,;V') \iff \int_0^T \Big[\sup\{\,|(\partial_t \boldsymbol{w}\,,\boldsymbol{\psi})| \mid \boldsymbol{\psi} \in V,\ \|\nabla\boldsymbol{\psi}\|_{2,\Omega} = 1\,\}\Big]^{4/3} \mathrm{d}t < \infty\,.$$

This is true if

$$\int_0^T \sup\Big\{\,|b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})|^{4/3} \mid \boldsymbol{\psi} \in V\,, \|\nabla\boldsymbol{\psi}\|_{2,\Omega} = 1\,\Big\}\,\mathrm{d}t < \infty,$$

and

$$\int_0^T \sup\Big\{\,|(\nabla \boldsymbol{w}\,,\nabla\boldsymbol{\psi})|^{4/3} \mid \boldsymbol{\psi} \in V\,, \|\nabla\boldsymbol{\psi}\|_{2,\Omega} = 1\,\Big\}\,\mathrm{d}t < \infty.$$

The second condition is true by the Cauchy-Schwarz inequality and the fact that $\nabla\boldsymbol{w} \in L^2(0,T\,;L^2(\Omega)) \subset L^{4/3}(0,T\,;L^2(\Omega))$. The first is also true since (4.14) implies

$$|b(\boldsymbol{w}\,,\boldsymbol{w}\,,\boldsymbol{\psi})|^{4/3} \le \|\boldsymbol{w}\|_{2,\Omega}^{8/3} \in L^1(0,T)\,. \qquad \square$$

So it is not clear that Leray-Hopf weak solutions satisfy the regularity criteria needed to be strong solutions. Alternatively, we could prove the existence of a strong solution. Specifically, we have the following theorem due to Leray [20]:

Theorem 4.7 (Local existence of strong solutions)**.** *Let $\boldsymbol{u}_0 \in V$. Then there exists $T^* \in (0,T]$ such that the initial-boundary value problem has a strong solution in Ω_{T^*}.*

Proof. The proof requires knowledge of solutions to the linear steady-state Navier-Stokes equations, so we provide a sketch, referring the reader to Galdi [10] for the details. Consider the Galerkin approximation (3.5) in the proof of Theorem 3.1. It can be shown[6] that the set of basis functions $\{\boldsymbol{\psi}_i\}_{i=1}^\infty \subset H$ can be taken as the eigenfunctions of

$$-\Delta\boldsymbol{\psi}_i + \nabla p_i = \lambda_i \boldsymbol{\psi}_{\boldsymbol{i}}\,, \quad \operatorname{div}\boldsymbol{\psi}_i = 0\,, \quad \boldsymbol{\psi}_i|_{\partial\Omega} = 0.$$

[6] See Galdi [10] Lemma 5.2.

Multiplying (3.6) by $\lambda_j f_{j,m}(t)$ and summing over $j = 1, \dots, m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^2 + \|P\Delta \boldsymbol{u}_m\|_{2,\Omega}^2 = b(\boldsymbol{u}_m\,, \boldsymbol{u}_m\,, P\Delta \boldsymbol{u}_m), \tag{4.21}$$

where P is the orthogonal projection of $L^2(\Omega)$ onto H. It can be shown[7] that we can estimate the right-hand side using the generalized Hölder inequality (2.2) and Sobolev embedding (B.10) by

$$\begin{aligned} |b(\boldsymbol{u}_m\,, \boldsymbol{u}_m\,, P\Delta \boldsymbol{u}_m)| &\le \|\boldsymbol{u}_m\|_{6,\Omega} \cdot \|\nabla \boldsymbol{u}_m\|_{3,\Omega} \cdot \|P\Delta \boldsymbol{u}_m\|_{2,\Omega} \\ &\le C\,\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^{\frac{3}{2}} \cdot \|P\Delta \boldsymbol{u}_m\|_{2,\Omega}^{\frac{3}{2}}\,. \end{aligned}$$

Apply Young's inequality (B.8) with

$$a = C_1\,\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^{\frac{3}{2}}\,, \quad b = C_2\,\|P\Delta \boldsymbol{u}_m\|_{2,\Omega}^{\frac{3}{2}}\,, \quad p = 4\,,\, p' = 4/3\,, \quad C_1 C_2 = C.$$

Then

$$|b(\boldsymbol{u}_m\,, \boldsymbol{u}_m\,, P\Delta \boldsymbol{u}_m)| \le C_3\,\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^6 + C_4\,\|P\Delta \boldsymbol{u}_m\|_{2,\Omega}^2\,.$$

Choose C_1 and C_2 such that $C_4 = 1$, and plug this into (4.21) to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^2 \le C\,\|\nabla \boldsymbol{u}_m\|_{2,\Omega}^6\,,$$

for some constant C.

Let $y_m(t) = \|\nabla \boldsymbol{u}_m\|_{2,\Omega}^2$. Then

$$\begin{aligned} &\frac{\mathrm{d}y_m}{\mathrm{d}t} \le C y_m^3 \\ \implies &\int_0^t \frac{\mathrm{d}y_m}{y_m^3} = \frac{1}{2y_m^2(0)} - \frac{1}{2y_m^2(t)} \le Ct. \end{aligned}$$

Thus, there exists a constant C such that

$$y_m^2(t) \le \frac{y_m^2(0)}{C - t \cdot y_m^2(0)}\,.$$

Hence,

$$\|\nabla \boldsymbol{u}_m(t)\|_{2,\Omega}^4 \le \frac{\|\nabla \boldsymbol{u}_0\|_{2,\Omega}^4}{C - t\,\|\nabla \boldsymbol{u}_0\|_{2,\Omega}^4}\,. \tag{4.22}$$

This estimate shows that the sequence $\nabla \boldsymbol{u}_m$ is bounded in $L^\infty(0, T_1 - \varepsilon\,; L^2(\Omega))$ for every $\varepsilon \in (0, T_1)$, where $T_1 = C/\|\nabla \boldsymbol{u}_0\|_{2,\Omega}^4$. Along with estimates (3.11) and (3.12),

[7] See Galdi [10] Lemma 5.3 and 5.4.

this allows us to conclude the result. From (4.22), we see that T^* satisfies

$$T^* \geq C/\|\nabla \boldsymbol{u}_0\|_{2,\Omega}^4 \,. \qquad \square$$

We can thus say that there exists a strong solution $\boldsymbol{u}$ to the initial-boundary value problem in the interval $[0, T^*)$ and that in this time period the solution is unique. If we can show that this strong solution in fact exists for all time, then we are done. In other words, the problem of uniqueness is translated into the following problem of regularity: do strong solutions to the initial-boundary value problem develop singularities in a finite time? This is, quite literally[8], the million-dollar question. The problem is open.

It is known that a strong solution is in fact at least as smooth as the initial data is. In particular, if we assume $\boldsymbol{u}_0 \in C^\infty(\Omega)$, as we shall from now, it can be shown that a strong solution is smooth. The technique used to do this is known as "bootstrapping", and the basic idea is as follows. We first show that our solution $\boldsymbol{u}$ is equal almost everywhere to some function $\boldsymbol{v}$. We then show that $\boldsymbol{v}$ has more regularity than was assumed for $\boldsymbol{u}$. From this, we deduce that $\boldsymbol{u}$ must have more regularity, and thus $\boldsymbol{v}$ must have more regularity, and so on. This is the other reason why strong solutions are so important. For the details, we refer the reader to Galdi [10] Ch.V or Temam [30] Theorem III.3.8. For a nice explanation of the bootstrap technique, see Tao [27].

Theorem 4.7 implies that the first instant that blow-up might occur is at

$$T^* \geq C/\|\nabla \boldsymbol{u}_0\|_{2,\Omega}^2.$$

From this, we deduce that a necessary condition for blow-up is

$$\limsup_{t \nearrow T^*} \|\nabla \boldsymbol{u}\|_{2,\Omega} \to \infty. \tag{4.23}$$

Indeed, suppose that this were not the case. Then there exists C such that

$$\|\nabla \boldsymbol{u}(t)\|_{2,\Omega} \leq C$$

for all t sufficiently close to T^*. Theorem 4.7 then implies that there exists a strong solution $\tilde{\boldsymbol{u}}$ starting at time $t = T^* - \varepsilon$, with initial velocity $\tilde{\boldsymbol{u}}_0 = \boldsymbol{u}(T^* - \varepsilon)$ satisfying $\|\nabla \tilde{\boldsymbol{u}}_0\|_{2,\Omega} \leq C$. This upper bound establishes a lower bound on the time from $T^* - \varepsilon$ that blow-up can occur, independent of ε. But this contradicts the requirement that blow-up occurs at T^*.

A Leray-Hopf weak solution $\boldsymbol{u}$, which exists for all time, necessarily coincides with the strong solution on $[0, T^*)$ by Theorem 4.5. Since $\boldsymbol{u} \in L^2(0, T\,; V)$ for all $T > 0$, then (4.23) implies that the set of all times that blow-up occurs has Lebesgue

[8]The Clay Mathematics Institute lists this problem as one of seven Millennium problems [6]; a solution, either in the affirmative or in the negative, is rewarded with a prize of $1 million.

measure zero. While this estimate is a start, there are two things which we wish to improve on. Firstly, though it gives an indication of *when* blow-up might occur, it gives no information on *where* this might happen. That is, we wish to classify the space-time singularities, not just the time singularities. Secondly, the Lebesgue measure suffers from the obvious deficiency that it does not give a good indication of the size of "small" sets. For example, a curve and a plane both have Lebesgue measure zero in $\mathbb{R}^3$, though one is obviously "smaller" than the other. These two points naturally lead us to the concept of *local regularity* and the Hausdorff measure.

5 Local Regularity Theory

5.1 Hausdorff Measure

Definition 5.1. Let $0 \leq \alpha < \infty$, $0 < \delta \leq \infty$, $E \subset \mathbb{R}^n$. Define

$$\mathcal{H}^\alpha_\delta(E) \coloneqq \inf \left\{ \sum_{k=1}^{\infty} (\operatorname{diam} F_k)^\alpha \mid E \subset \bigcup_{k=1}^{\infty} F_k,\ \operatorname{diam} F_k \leq \delta\ \forall k \right\},$$

where $\operatorname{diam} A = \sup\{\, |x-y| \mid x, y \in A \,\}$. Notice that $\mathcal{H}^\alpha_{\delta_1}(E) > \mathcal{H}^\alpha_{\delta_2}(E)$ if $\delta_1 < \delta_2$. Thus, $\mathcal{H}^\alpha_\delta(E)$ increases as δ decreases, and this allows us to define

$$\mathcal{H}^\alpha(E) \coloneqq \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E) \in [0, \infty].$$

$\mathcal{H}^\alpha$ is the α-*dimensional Hausdorff measure* on $\mathbb{R}^n$.

The motivation for this definition is as follows. Notice that the measure of a set E scales like its dimension. For example, if E is a sphere in $\mathbb{R}^n$, then the measure, or volume, of E scales like $V \sim r^n$. This is the basic property that the definition attempts to capture. We take the limit as $\delta \to 0$ in order to capture the small-scale behaviour of the set.

For our purposes, we shall be interested in the *parabolic Hausdorff measure*, defined in an entirely analogous manner, but with parabolic cylinders replacing the sets F_k. To state the definition, it is convenient to introduce the notation:

- $z = (x, t) \in \mathbb{R}^3 \times \mathbb{R}$.
- $B(x_0, r) = \{\, x \in \mathbb{R}^3 \mid |x - x_0| < r \,\}$ is the ball of radius r at x_0.
- $Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0)$ is the parabolic cylinder at $z_0 = (x_0, t_0)$.
- $Q^*(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0 + r^2)$.

Definition 5.2. Let $0 \leq \alpha < \infty$, $0 < \delta \leq \infty$, $E \subset \mathbb{R}^3 \times \mathbb{R}$. Define

$$\mathcal{P}^\alpha_\delta(E) = \inf \left\{ \sum_{k=1}^{\infty} r_k^\alpha \mid E \subset \bigcup_{k=1}^{\infty} Q^*(z_k, r_k),\ r_k < \delta \right\}.$$

The α-*dimensional parabolic Hausdorff measure* $\mathcal{P}^\alpha$ on $\mathbb{R}^3 \times \mathbb{R}$ is defined as

$$\mathcal{P}^\alpha(E) = \lim_{\delta \to 0} \mathcal{P}^\alpha_\delta(E).$$

If $r < 1$, then $r^2 \leq r$, so any parabolic cylinder $Q(z_0, r)$ with $r < \delta \leq 1$ satisfies $\operatorname{diam} Q(z_0, r) \leq C\delta$. Thus, it follows that

$$\mathcal{H}^\alpha \leq C(\alpha)\, \mathcal{P}^\alpha.$$

Note that

$$\mathcal{P}^\alpha(E) = 0 \iff \forall\, \varepsilon > 0,\ \exists \bigcup_{k=1}^{\infty} Q^*(z_k, r_k) \supset E \text{ such that } \sum_{k=1}^{\infty} r_k^\alpha < \varepsilon. \tag{5.1}$$

5.2 Caffarelli-Kohn-Nirenberg Theorem

The basic idea of local regularity theory is to analyse the behaviour of solutions at a local level by dropping the boundary and initial conditions and focusing on the equations of motion. The goal is to show that a weak solution to the equations of motion in a canonical domain is smoother in subdomains. There are two cases to consider: the first is the case of *interior regularity*, while the second is the case of *boundary regularity*. We shall only consider the first.

Definition 5.3. Let $\boldsymbol{u}$ be a Leray-Hopf weak solution. A point $z_0 = (x_0, t_0) \subset \Omega_T$ is called *regular* if $\boldsymbol{u}$ is bounded on $Q(z_0, r)$ for all sufficiently small r. A point z_0 is *singular* if it is not regular.

Remark 5.4. *With higher regularity results, the "bounded" condition is equivalent to Hölder continuity.*

Let S be the set of all singular points in Ω_T. The term *partial regularity* refers to an estimate on the size of S. Partial regularity theory for the Navier-Stokes equations was studied by Scheffer [22],[23],[24] in a series of papers, in which he proved that there exists a weak solution whose singular set satisfies $\mathcal{H}^{5/3}(S) < \infty$. This was then improved upon by Caffarelli, Kohn and Nirenberg [3] in their famous paper where they proved, among other things, that $\mathcal{P}^1(S) = 0$, which is the optimal result to date. They proved this by introducing *suitable weak solutions*, which are the analogue of Leray-Hopf weak solutions in the local setting.

Definition 5.5. A pair $(\boldsymbol{u}, p)$ is called a *suitable weak solution* to the Navier-Stokes system in the space-time domain $\omega \times (t_1, t_2)$ if:

1. $\boldsymbol{u} \in L^\infty(t_1, t_2\,; L^2(\omega)) \cap L^2(t_1, t_2\,; W^{1,2}(\omega))$.
2. $p \in L^{3/2}(\omega \times (t_1, t_2))$.
3. The equation (1.2) is satisfied in the sense of distributions:

$$\iint_{\omega\times(t_1,t_2)} u_k \frac{\partial \phi}{\partial x_k}\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, \phi \in C_0^\infty(\omega \times (t_1, t_2)).$$

4. The equation (1.1) is satisfied in the sense of distributions:

$$\iint_{\omega\times(t_1,t_2)} \left\{ u_i \frac{\partial \varphi_i}{\partial t} + u_k \frac{\partial \varphi_i}{\partial x_k} u_i + u_i \frac{\partial^2 \varphi_i}{\partial x_k \partial x_k} \right\} \mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, \boldsymbol{\varphi} \in C_{0,0}^\infty(\omega\times(t_1, t_2)).$$

5. The *local energy inequality*

$$\int_\omega \phi(x,t)|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x + 2\int_{t_1}^t\int_\omega \phi(x,\tau)|\nabla\boldsymbol{u}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq \int_{t_1}^t\int_\omega \left\{\left(\frac{\partial\phi}{\partial t}+\Delta\phi\right)|\boldsymbol{u}(x,\tau)|^2 + \big(|\boldsymbol{u}|^2+2p\big)\boldsymbol{u}\cdot\nabla\phi\right\}\mathrm{d}x\,\mathrm{d}\tau$$

holds for almost all $t \in (t_1, t_2)$ and for every $\phi \in C_0^\infty(\omega\times(t_1,\infty))$ with $\phi \geq 0$.

In [3], Caffarelli *et al.* proved the global existence of suitable weak solutions as a subclass of Leray-Hopf weak solutions. The proof that they give is different from the Galerkin method presented in Chapter 3. Indeed, it is not known whether the solutions obtained from the Galerkin method are suitable. The method of their proof, based upon a regularization of the nonlinear term, is similar to both Leray's original proof [20] and the semi-discretization method used by Temam [30] to prove existence of Leray-Hopf weak solutions. We refer the reader to [3] Theorem A.1 and Seregin [25] Theorem 6.11 for the proof.

The main results of Caffarelli *et al.* were regarding the smoothness of suitable weak solutions. In particular, they proved the following theorem (we give a rescaled version due to Seregin [25], Lemma 6.1):

Theorem 5.6. *There exists a constant $\varepsilon_0 > 0$ with the following property. Suppose $(\boldsymbol{u}, p)$ is a suitable weak solution in $Q(z_0, r)$ satisfying*

$$\frac{1}{r^2}\iint_{Q(z_0,r)}\left(|\boldsymbol{u}|^3+|p|^{\frac{3}{2}}\right)\mathrm{d}x\,\mathrm{d}t < \varepsilon_0. \tag{5.2}$$

Then $\nabla^{k-1}\boldsymbol{u}$ is Hölder continuous in $\overline{Q}(z_0, r/2)$ for all $k \in \mathbb{N}$. That is, there exists $\alpha \in (0,1]$ and constants C_k such that

$$|\nabla^{k-1}\boldsymbol{u}(z) - \nabla^{k-1}\boldsymbol{u}(z')| \leq C_k\,\|z-z'\|_{\mathrm{par}}^\alpha\,,\quad \forall\, k\in\mathbb{N},\ z,z'\in\overline{Q}(z_0,r/2),$$

where $\|z\|_{\mathrm{par}} = |x| + \sqrt{|t|}$.

Remark 5.7. *To see why we only get smoothness in the spatial variables, let $h(x)$ satisfy $\Delta h = 0$, and consider $\boldsymbol{u}(x,t) = a(t)\nabla h(x)$. Then $\boldsymbol{u}$ is a suitable weak solution even if $a(t)$ is only in $L^\infty(0,T)$. As a result, we cannot get any more smoothness in time. Note that $\boldsymbol{u}$ is smooth in space, since h satisfies Laplace's equation, and is thus smooth (since h is the real part of a holomorphic function).*

A consequence of this result is the following theorem (Proposition 2 in [3]):

Theorem 5.8. *There exists a constant $\varepsilon_1 > 0$ with the following property. Suppose $(\boldsymbol{u}, p)$ is a suitable weak solution near the point z_0 satisfying*

$$\limsup_{r \to 0} \frac{1}{r} \iint_{Q^*(z_0,r)} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}x \, \mathrm{d}t < \varepsilon_1. \tag{5.3}$$

Then z_0 is a regular point.

Remark 5.9. *In condition* (5.3) *we could also integrate over $Q(z_0, r)$.*

The proofs of these results are beyond the scope of this dissertation, but we shall provide some motivation for why we might expect them to be true.

The Navier-Stokes equations obey an important scaling property: if $(\boldsymbol{u}, p)$ is a solution to (1.1)-(1.4) in the canonical domain $B(0,1) \times (0,1]$, then $(\boldsymbol{u}_\lambda, p_\lambda)$ is a solution in $B(0,1/\lambda) \times [0,1/\lambda^2]$, where $\lambda > 0$ and

$$\boldsymbol{u}_\lambda : B(0,1/\lambda) \times \left(0, 1/\lambda^2\right] \to \mathbb{R}^3, \qquad \boldsymbol{u}_\lambda(x,t) = \lambda \boldsymbol{u}\left(\lambda x, \lambda^2 t\right) \tag{5.4}$$

$$p_\lambda : B(0,1/\lambda) \times \left(0, 1/\lambda^2\right] \to \mathbb{R}, \qquad p_\lambda(x,t) = \lambda^2 p\left(\lambda x, \lambda^2 t\right). \tag{5.5}$$

Consider the case when $\lambda \ll 1$. The mapping from $\boldsymbol{u}$ to $\boldsymbol{u}_\lambda$ "zooms in" on the origin, since events occurring in the "small" domain $B(0,\varepsilon) \times (0, \varepsilon^2]$ now occur in the larger domain $B(0, \varepsilon/\lambda) \times (0, \varepsilon^2/\lambda^2]$.

One obvious way which we might try to control $\boldsymbol{u}$ is through the kinetic energy $E(t) = \frac{1}{2}\rho \|\boldsymbol{u}(\cdot,t)\|_{2,\Omega}^2$. The problem, however, is that the velocity can seemingly blow up in finite time with $E(t)$ remaining bounded, as we showed in Chapter 1. This is because $E(t)$ scales like $1/\lambda$, so that as we shift down to smaller scales, the control over $\boldsymbol{u}$ gets increasingly worse. In this sense, the energy is a *supercritical* quantity. This is in contrast with *subcritical* quantities, which are "small at small scales", and *critical* quantities, which are invariant under scaling.

Following Caffarelli *et al.* [3], we formalize this by assigning each quantity a "dimension" that indicates their behaviour at small scales:

$$\begin{aligned} \dim x &= 1, & \dim t &= 2, & \dim \boldsymbol{u} &= -1, \\ \dim \nabla &= -1, & \dim \partial_t &= -2, & \dim p &= -2. \end{aligned}$$

Note that since $\dim t = 2$, it is natural to work with parabolic cylinders.

Quantities with which we might wish to control $\boldsymbol{u}$ include

$$\text{Energy}: \qquad \dim \int_\Omega |\boldsymbol{u}(x,t)|^2 \, \mathrm{d}x = 1.$$

$$L^3 \text{ norm}: \qquad \dim \int_\Omega |\boldsymbol{u}(x,t)|^3 \, \mathrm{d}x = 0.$$

$$\text{Dissipation :} \quad \dim \int_\Omega |\nabla \boldsymbol{u}(x,t)|^2 \, \mathrm{d}x = -1.$$

Note that the quantities $\boldsymbol{u}$, $\nabla \boldsymbol{u}$ and $\partial_t \boldsymbol{u}$ all have negative dimension. Thus, if we wish to establish regularity of $\boldsymbol{u}$, we need to control quantities with negative dimension or impose that a quantity with dimension zero be "small". This is essentially the idea behind Theorem 5.6. The quantity (5.2) has dimension zero, and Theorem 5.6 says that smallness of this is enough to establish regularity. Similarly, Theorem 5.8 says that smallness at small scales of the critical quantity in condition (5.3) is enough to establish regularity.

Condition (5.3) is reminiscent of the Lebesgue differentiation theorem from classical measure theory (see Jones [14] §15). The *Lebesgue set* of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$\mathrm{Leb}(f) = \left\{ x \in \mathbb{R}^n \mid \exists\, \alpha \in \mathbb{R} \text{ such that } \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, \mathrm{d}y = \alpha \right\}.$$

Then the Lebesgue differentiation theorem states that $\mathbb{R}^n \backslash \mathrm{Leb}(f)$ has Lebesgue measure zero.

Now, in $\mathbb{R}^3 \times \mathbb{R}$ we have that $|Q^*(z_0, r)| = Cr^5$. It follows that (5.3) is a much weaker condition than the one needed for a point to belong to the Lebesgue set of a function. As a result, we might expect that "more" points satisfy (5.3), and thus the set of points that do not satisfy this, namely the singular points, is "smaller" than any set of Lebesgue measure zero. More precisely, we have the famous Caffarelli-Kohn-Nirenberg Theorem (Theorem B in [3]):

Theorem 5.10 (Caffarelli-Kohn-Nirenberg Theorem)**.**
For any suitable weak solution of the Navier-Stokes equations on a bounded open set in space-time, the singular set S satisfies $\mathcal{P}^1(S) = 0$.

To prove this, we will need the following result, which is the analogue of the classical Vitali covering lemma:

Lemma 5.11. *Given a family of parabolic cylinders $\mathcal{F} = \{Q^*(z,r)\}$, there exists a finite or countable subfamily $\mathcal{G} = \{Q^*(z_i, r_i)\}_{i=1}^{\infty}$ such that*

$$Q^*(z_i, r_i) \cap Q^*(z_j, r_j) = \varnothing,$$

$$\forall\, Q^*(z,r) \in \mathcal{F},\ \exists\, j \in \mathbb{N} \text{ such that } Q^*(z,r) \subset Q^*(z_j, 5r_j).$$

Proof. We adapt the proof in [5] §1.5.1 (see also [3] Lemma 6.1). Let

$$R = \sup \{ r \mid Q^*(z,r) \in \mathcal{F} \},$$

$$\mathcal{F}_k = \{ Q^*(z,r) \in \mathcal{F} \mid R/2^k < r \leq R/2^{k-1} \}.$$

Let $\mathcal{G}_1$ be any maximal disjoint subcollection of parabolic cylinders in $\mathcal{F}_1$. Proceed inductively by choosing $\mathcal{G}_k$ to be any maximal disjoint subcollection of

$$\mathcal{J}_k = \left\{ Q^*(z,r) \in \mathcal{F}_k \mid Q^*(z,r) \cap Q^*(z',r') = \varnothing,\ \forall\, Q^*(z',r') \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \right\}.$$

Define $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$. Then $\mathcal{G} \subset \mathcal{F}$ is disjoint and countable. Take $Q^*(z,r) \in \mathcal{F}$. We may assume $Q^*(z,r) \notin \mathcal{G}$, since otherwise the result is obvious. Note that $\{\, (R/2^j, R/2^{j-1}]\,\}_{j=1}^{\infty}$ is a partition of $(0,R]$, so that $Q^*(z,r) \in \mathcal{F}_k$ for some $k \in \mathbb{N}$.

If $Q^*(z,r) \notin \mathcal{J}_k$, then $Q^*(z,r) \cap Q^*(z',r') \neq \varnothing$ for some $Q^*(z',r') \in \cup_{j=1}^{k-1}\mathcal{G}_j$. If $Q^*(z,r) \in \mathcal{J}_k$, then $Q^*(z,r)\cap Q^*(z',r') \neq \varnothing$ for some $Q^*(z',r') \in \mathcal{G}_k$, since otherwise the maximality of $\mathcal{G}_k$ would imply that $Q^*(z,r) \in \mathcal{G}_k \subset \mathcal{G}$. Either way, we can find a parabolic cylinder $Q^*(z',r')$ such that

$$Q^*(z',r') \in \bigcup_{j=1}^{k} \mathcal{G}_j, \tag{5.6}$$

$$Q^*(z,r) \cap Q^*(z',r') \neq \varnothing. \tag{5.7}$$

Then (5.6) implies that $r' > R/2^k$. Since $Q^*(z,r) \in \mathcal{F}_k$, we know that $r \leq R/2^{k-1}$, so that $r < 2r'$. This, (5.7) and the triangle inequality imply the result. □

Proof (of Theorem 5.10). The proof, following [3], is based on a covering argument. Let $(\boldsymbol{u}, p)$ be a suitable weak solution in $\omega \times (t_1,t_2)$, D a neighbourhood of S and $\delta > 0$. By Theorem 5.8,

$$z \in S \implies \limsup_{r\to 0} \frac{1}{r} \iint_{Q^*(z,r)} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \geq \varepsilon_1.$$

Thus, given $z \in S$, we can find $r < \delta$ such that $Q^*(z,r) \subset D$, and

$$\frac{1}{r} \iint_{Q^*(z,r)} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \geq \varepsilon_1. \tag{5.8}$$

We do this for each $z \in S$, and then select a countable disjoint subfamily of parabolic cylinders $\{Q^*(z_i,r_i)\}_{i=1}^{\infty}$ such that

$$S \subset \bigcup_{i=1}^{\infty} Q^*(z_i, 5r_i).$$

From (5.8), we see that

$$\sum_{i=1}^{\infty} r_i \;\leq\; \frac{1}{\varepsilon_1} \sum_{i=1}^{\infty} \iint_{Q^*(z_i,r_i)} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \;\leq\; \frac{1}{\varepsilon_1} \iint_D |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\Longrightarrow\ \mathcal{P}^1(S)\ \le\ \sum_{i=1}^{\infty} 5\,r_i\ \le\ \frac{5}{\varepsilon_1}\iint_D |\nabla \boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{5.9}$$

Since $\nabla \boldsymbol{u} \in L^2(t_1, t_2\,; L^2(\omega)) \subset L^2(D)$, then $\sum_i r_i < \infty$. Thus, if $m(S)$ denotes the Lebesgue measure of S, we find that

$$m(S)\ \le\ C\sum_{i=1}^{\infty}(5r_i)^5\ \le\ C\delta^4\sum_{i=1}^{\infty} r_i \to 0, \quad \text{as } \delta \to 0.$$

Since (5.9) holds for every neighbourhood D of S, and $m(S) = 0$, we can make the right-hand side of (5.9) arbitrarily small by choosing D such that $m(D\backslash S)$ is sufficiently small, and the result follows from (5.1). □

6 Axially Symmetric Flow Without Swirl

Our goal in this chapter is to describe how one can use the results discussed earlier to analyse regularity for the special case of axially symmetric flow without swirl.

6.1 Formulation

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be an orthonormal basis for Cartesian coordinates (x_1, x_2, x_3), and $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_3\}$ an orthonormal basis for cylindrical polar coordinates (r, θ, x_3) with

$$\boldsymbol{e}_r = \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2, \quad \boldsymbol{e}_\theta = -\sin\theta\, \boldsymbol{e}_1 + \cos\theta\, \boldsymbol{e}_2.$$

Then, for any vector $\boldsymbol{u} \in \mathbb{R}^3$, we define the radial u_r and axial u_θ components via

$$\boldsymbol{u} = u_k \boldsymbol{e}_k = u_1 \boldsymbol{e}_1 + u_2 \boldsymbol{e}_2 + u_3 \boldsymbol{e}_3 = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_3 \boldsymbol{e}_3.$$

Suppose that the flow is *axially symmetric* i.e. the velocity and pressure are independent of θ. Assume further that there is no *swirl* i.e. $u_\theta(r, z) \equiv 0$.

It is natural to expect better behaviour of solutions when we have this additional symmetry. Indeed, Ukhovskii and Iudovich [31] and, independently, Ladyzhenskaya [16] proved global existence of smooth solutions for the case $\Omega = \mathbb{R}^3$, see also Leonardi *et al.* [19]. However, it is not known whether the same result holds for the case $\Omega \subset \mathbb{R}^3$ bounded. For the half-space $\Omega = \mathbb{R}^3_+$, Kang [15] showed that there are no singular points, except possibly at the origin. A stronger result was given by Seregin and Sverák ([26], Theorem 1.3), namely of *local* regularity: for any domain Ω, there are no singular points away from the boundary $\partial\Omega$. We now show how one might prove this result.

Firstly, we know that there exists a Leray-Hopf weak solution that is also a suitable weak solution in Ω_T. By Theorem 4.7, there exists a strong solution defined on a time interval $[0, T^*)$. Since $\boldsymbol{u}_0$ is smooth, this strong solution is smooth in both the spatial and time variables. By Theorem 4.5, it is unique in the class of Leray-Hopf weak solutions, and thus coincides with the suitable weak solution on $[0, T^*)$.

By Theorem 5.10 and the fact that the flow is axially symmetric, any singular points must lie on the axis symmetry. Otherwise, a ring of singular points would exist, which would imply that $\mathcal{P}^1(S) \neq 0$. Seek a contradiction by supposing $x_0 = (0, 0, x_{0,3}) \in \Omega$ is a point on the axis of symmetry with $z_0 = (x_0, T_0)$ a singular point. There are two cases to consider: $T_0 = T^*$ and $T_0 > T^*$.

6.2 Case I: $T_0 = T^*$

Suppose $T_0 = T^*$. Take $r > 0$ sufficiently small such that $B(x_0, r) \subset \Omega$ and $r^2 < T_0$, and consider the domain $B(x_0, r) \times [T_0 - r^2, T_0]$. Using the scaling invariance of the

Navier-Stokes equations (cf. §5.2), we can translate and scale this domain to the canonical domain $B(0,1) \times [-1,0]$, with $z_0 = 0$ the singular point and $\boldsymbol{u}$ smooth in $Q_\varepsilon := B(0,1) \times [-1, -\varepsilon^2]$ for every $\varepsilon \in (0,1]$.

Lemma 6.1. *There exists $\alpha \in (0,1)$ and $\delta' > 0$ such that $\boldsymbol{u}$ is Hölder continuous on*

$$\Big\{ z \in B(0,1) \times \{t = 0\} \;\Big|\; |x| \in [\alpha - \delta', \alpha + \delta'] \Big\}.$$

Proof (See Figure 1). Since all the singular points lie on the x_3 axis, this is equivalent to the claim that there exists $\alpha' \in (0,1)$ such that $z = (0,0,\pm\alpha',0)$ is a regular point. But the existence of such an α' is immediate, since $\mathcal{P}^1(S') = 0$, where

$$S' = \{ (0,0,\beta,0) \in S \mid (0,0,-\beta,0) \in S \}.$$

□

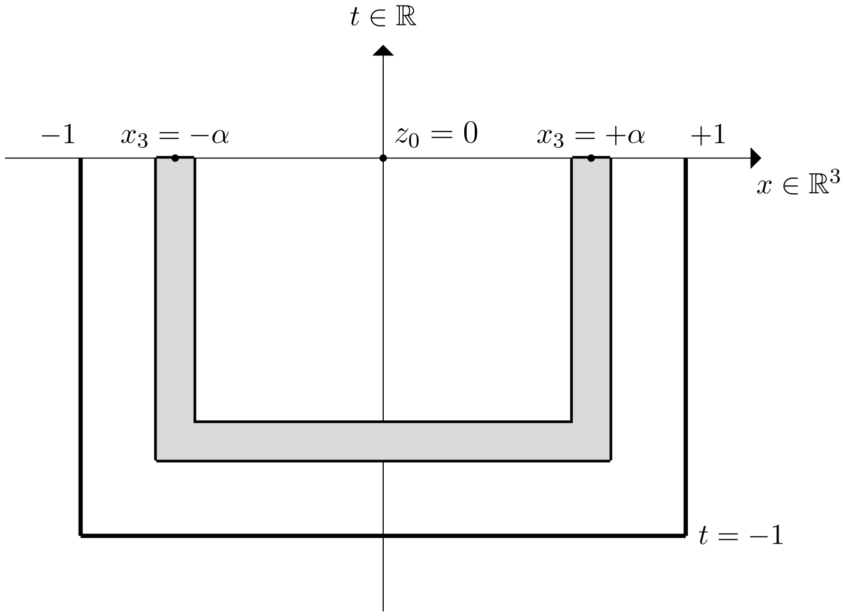

Figure 1: *$\boldsymbol{u}$ is Hölder continuous on the boundary of the shaded region and smooth in the interior of the parabolic cylinder $Q(0,1)$.*

Now consider the domain $B(0, \alpha+\delta') \times [-(\alpha+\delta')^2, 0]$. Again, by rescaling, we find that

$$\boldsymbol{u} \text{ is a suitable weak solution in } B(0,1) \times [-1,0]. \tag{6.1}$$

$$\boldsymbol{u} \text{ is smooth in } Q_\varepsilon \text{ for every } \varepsilon \in (0,1]. \tag{6.2}$$

$$\boldsymbol{u} \text{ is Hölder continuous on } \partial B(0,1) \times [-1,0]. \tag{6.3}$$

6.2.1 Vorticity Equation

Note that the following formal calculations are allowed, since $\boldsymbol{u}$ is smooth in Q_ε for $\varepsilon \in (0,1]$.

Define the *vorticity* $\boldsymbol{w}$ by $\boldsymbol{w} := \nabla \times \boldsymbol{u}$. Taking the curl of the momentum equation, we obtain the *vorticity transport equation*:

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{w} = (\boldsymbol{w} \cdot \nabla)\, \boldsymbol{u} + \Delta \boldsymbol{w} \tag{6.4}$$

Since $u_\theta = 0$, then $\boldsymbol{w} = w_\theta \boldsymbol{e}_\theta = (\partial_z u_r - \partial_r u_z)\, e_\theta$ and (6.4) reduces to a scalar equation and may be written in component form as

$$\frac{\partial w_\theta}{\partial t} + u_r \frac{\partial w_\theta}{\partial r} + u_z \frac{\partial w_\theta}{\partial z} = \frac{1}{r} w_\theta u_r + \frac{\partial^2 w_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial w_\theta}{\partial r} + \frac{\partial^2 w_\theta}{\partial z^2} - \frac{1}{r^2} w_\theta.$$

Define $\eta := w_\theta / r$. Substituting this into the above equation gives

$$\frac{\partial \eta}{\partial t} + u_r \frac{\partial \eta}{\partial r} + u_z \frac{\partial \eta}{\partial z} - \left[\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2} + \frac{3}{r}\frac{\partial \eta}{\partial r}\right]. \tag{6.5}$$

We now extend η to a function of *five* spatial variables, say x_1, x_2, x_3, x_4, z, and time. Let $\eta(r,z,t) = U(x_1,x_2,x_3,x_4,z,t)$, with

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_4 &= r\sin\theta_1\sin\theta_2\sin\theta_3, \end{aligned}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $0 \le \theta_1, \theta_2 \le \pi$ and $0 \le \theta_3 \le 2\pi$.

Using the expression for the gradient and Laplace operators in 4-d spherical coordinates, we find that $U(x_1,x_2,x_3,x_4,z,t)$ satisfies the equation

$$\partial_t U + \boldsymbol{b} \cdot \nabla_5 U - \Delta_5 U = 0, \tag{6.6}$$

in $Q_{5,\varepsilon} := B_5(0,1) \times (-1, -\varepsilon^2) \subset \mathbb{R}^5 \times \mathbb{R}$.

Here, ∇_5 and Δ_5 are the usual gradient and Laplace operators with respect to

Cartesian coordinates in $\mathbb{R}^5$, and $\boldsymbol{b} \in \mathbb{R}^5$ is given by

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{b}(x_1, x_2, x_3, x_4, z, t) = b_i \boldsymbol{e}_i \\ b_i &= \frac{u_r(r,z,t)}{r} x_i \,, \quad i = 1,2,3,4 \\ b_5 &= u_z(r,z,t). \end{aligned} \tag{6.7}$$

We have thus obtained a 5-dimensional heat equation with drift, an equation to which we can apply results from the theory of parabolic equations.

6.2.2 Maximum Principle Argument

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $T > 0$. Suppose that $F \in C(\overline{\Omega_T}) \cap C_1^2(\Omega_T)$ satisfies*

$$\partial_t F + \boldsymbol{B} \cdot \nabla F - \Delta F \leq 0 \quad \textit{in } \Omega_T, \tag{6.8}$$

where $\boldsymbol{B}$ is continuous. Then

$$\sup_{\overline{\Omega_T}} F \leq \sup_{\partial^* \Omega_T} F,$$

where $\partial^* \Omega_T = \big(\overline{\Omega} \times \{t = 0\}\big) \cup (\partial\Omega \times [0,T])$ is the parabolic boundary of Ω_T.

Proof (Evans [4], §7.1.4). Since $F \in C(\overline{\Omega_T})$, the supremum exists and is attained at some point. We proceed by contradiction. Assume first that

$$\partial_t F + \boldsymbol{B} \cdot \nabla F - \Delta F < 0 \quad \text{in } \Omega_T. \tag{6.9}$$

Suppose there exists (x_0, t_0) with $x_0 \in \Omega$ and $0 < t_0 \leq T$ such that

$$F(x_0, t_0) = \sup_{\overline{\Omega_T}} F(x,t).$$

Then

$$\begin{aligned} \partial_t F(x_0, t_0) &= 0, \quad \text{if } t_0 < T; \\ \partial_t F(x_0, t_0) &\geq 0, \quad \text{if } t_0 = T; \\ \nabla F(x_0, t_0) = 0 \quad &\text{and} \quad \Delta F(x_0, t_0) \leq 0, \end{aligned}$$

which contradicts (6.9). Thus, the maximum is attained on $\partial^* \Omega_T$.

Now assume (6.8) holds. Let $F^\varepsilon(x,t) \coloneqq F(x,t) - \varepsilon t$, with $\varepsilon > 0$. Then

$$\partial_t F^\varepsilon + \boldsymbol{B} \cdot \nabla F^\varepsilon - \Delta F^\varepsilon < 0 \quad \text{in } \Omega_T,$$

so F^ε attains its maximum on $\partial^*\Omega_T$. Let $\varepsilon \to 0$ to conclude the result. $\square$

Return now to equation (6.6). To apply Theorem 6.2, we need to verify that U and $\boldsymbol{b}$ satisfy

$$U \in C_1^2(Q_{5,\varepsilon}) \cap C(\overline{Q_{5,\varepsilon}}), \tag{6.10}$$

$$\boldsymbol{b} \in C(Q_{5,\varepsilon}). \tag{6.11}$$

Note that

$$0 = \operatorname{div} \boldsymbol{u} = \frac{1}{r} u_r + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}$$

$$\implies \frac{1}{r} u_r = -\frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z}.$$

Since the right-hand side is in $C(Q_{5,\varepsilon})$, then so is the left-hand side. It follows then from (6.7) that (6.11) holds.

Assumption (6.10) is also true since $\boldsymbol{w}$ is smooth in Q_ε, so that

$$\begin{aligned}\nabla \boldsymbol{w} &= \frac{\partial \boldsymbol{w}}{\partial r} \otimes \boldsymbol{e}_r + \frac{1}{r}\frac{\partial \boldsymbol{w}}{\partial \theta} \otimes \boldsymbol{e}_\theta + \frac{\partial \boldsymbol{w}}{\partial z} \otimes \boldsymbol{e}_z \\ &= \frac{\partial w_\theta}{\partial r} \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r - \eta \boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \frac{\partial w_\theta}{\partial z} \boldsymbol{e}_\theta \otimes \boldsymbol{e}_z\end{aligned}$$

is smooth in Q_ε, and thus η is smooth in Q_ε. Here, $\otimes$ is the tensor product. Thus, U, $\partial_t U$, $\nabla_5 U$ and $\Delta_5 U$ all exist and are continuous in $Q_{5,\varepsilon}$ so that (6.10) holds.

Now we can apply Theorem 6.2 to find that U is bounded on $\overline{Q_{5,\varepsilon}}$ by its values on

$$\partial^* Q_{5,\varepsilon} = (\partial B_5(0,1) \times [-1, -\varepsilon^2]) \cup (\overline{B}_5(0,1) \times \{t = -1\}),$$

see Figure 2. Since U is bounded on $\overline{B}_5(0,1) \times \{t = -1\}$ and on $\partial B_5(0,1) \times [-1, 0]$, we see that U is bounded in $\overline{Q_{5,\varepsilon}}$. Thus, η and hence $\boldsymbol{w}$ are bounded in Q_ε, independent of ε.

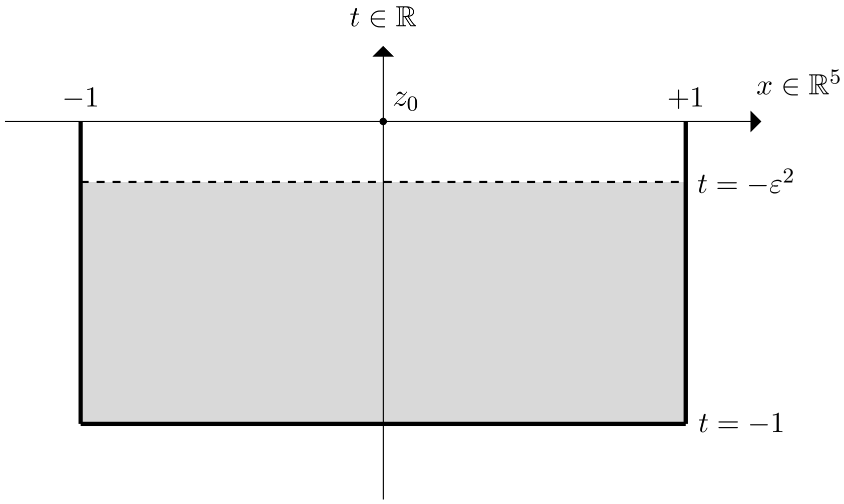

Figure 2: *The picture for the maximum principle argument. U is bounded in the shaded region by its values on the parabolic boundary indicated by thick lines.*

To bound $\boldsymbol{u}$, we will need the following variant of *Korn's inequality*:

Lemma 6.3. *Let the matrix $\boldsymbol{\epsilon}(\boldsymbol{v})$ be given by*

$$\epsilon_{i,j}(\boldsymbol{v}) = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}. \tag{6.12}$$

Let $x_0 \in \mathbb{R}^3$ and $\operatorname{div} \boldsymbol{v} = 0$. Then there exists a constant C such that:

(i) $\|\nabla \boldsymbol{v}\|_{2,B(x_0,r)} \leq C \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)}, \quad \forall \boldsymbol{v} \in W_0^{1,2}(B(x_0,r))$*;*

(ii) $\|\nabla \boldsymbol{v}\|_{2,B(x_0,r/2)} \leq C\Big[\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)} + \|\boldsymbol{v}\|_{2,B(x_0,r)}\Big], \quad \forall \boldsymbol{v} \in W^{1,2}(B(x_0,r)).$

Proof. By density arguments, it is enough to prove this for $\boldsymbol{v} \in C_{0,0}^{\infty}(B(x_0,r))$.
(i) We apply integration by parts. Since $\boldsymbol{v}$ is divergence free and has compact support in $B(x_0,r)$,

$$\begin{aligned}
\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)}^2 &= \int_{B(x_0,r)} \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}x \\
&= 2\|\nabla \boldsymbol{v}\|_{2,B(x_0,r)}^2 - 2\int_{B(x_0,r)} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \,\mathrm{d}x \\
&= 2\|\nabla \boldsymbol{v}\|_{2,B(x_0,r)}^2 - 2\int_{B(x_0,r)} \frac{\partial}{\partial x_i}\left(v_j \frac{\partial v_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(v_j \frac{\partial v_i}{\partial x_i}\right) + \frac{\partial v_j}{\partial x_j}\frac{\partial v_i}{\partial x_i} \,\mathrm{d}x \\
&= 2\|\nabla \boldsymbol{v}\|_{2,B(x_0,r)}^2 - 2\int_{\partial B(x_0,r)} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} \cdot \boldsymbol{n} \,\mathrm{d}x + 2\int_{B(x_0,r)} \frac{\partial}{\partial x_j}\left(v_j \operatorname{div} \boldsymbol{v}\right) \mathrm{d}x
\end{aligned}$$

$$- 2 \int_{B(x_0,r)} (\operatorname{div} \boldsymbol{v})^2 \, \mathrm{d}x$$

$$= 2 \|\nabla \boldsymbol{v}\|^2_{2,B(x_0,r)} .$$

So if $C \geq \frac{1}{2}$, then

$$\|\nabla \boldsymbol{v}\|^2_{2,B(x_0,r)} \leq C \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)}$$

(ii) Apply (i) to $\varphi \boldsymbol{v}$, where φ is a cut-off function i.e.

$$\begin{cases} \varphi \in C_0^\infty(B(x_0,r)) & \\ \varphi \equiv 1, & \text{in } B(x_0, r/2) \\ 0 \leq \varphi \leq 1, & \text{in } B(x_0, r) \end{cases}$$

Then

$$\|\nabla(\varphi \boldsymbol{v})\|_{2,B(x_0,r)} \leq C \|\boldsymbol{\epsilon}(\varphi \boldsymbol{v})\|_{2,B(x_0,r)} .$$

By the properties of φ,

$$\begin{aligned} \int_{B(x_0,r)} |\nabla(\varphi \boldsymbol{v})|^2 \, \mathrm{d}x &= \int_{B(x_0,r)} |\varphi \nabla \boldsymbol{v} + \boldsymbol{v}^T \nabla \varphi|^2 \, \mathrm{d}x \\ &\geq \int_{B(x_0,r/2)} |\varphi \nabla \boldsymbol{v} + \boldsymbol{v}^T \nabla \varphi|^2 \, \mathrm{d}x \\ &= \int_{B(x_0,r/2)} |\nabla \boldsymbol{v}|^2 \, \mathrm{d}x. \end{aligned}$$

Now, we have that

$$\epsilon_{i,j}(\varphi \boldsymbol{v}) = \varphi \cdot \epsilon_{i,j}(\boldsymbol{v}) + \frac{\partial \varphi}{\partial x_i} v_j - \frac{\partial \varphi}{\partial x_j} v_i.$$

By the equivalence of norms in finite dimensions and the fact that $\varphi \in C_0^\infty(B(x_0,r))$, we have that

$$\begin{aligned} |\epsilon_{i,j}(\varphi \boldsymbol{v})| &\leq |\varphi| \cdot |\epsilon_{i,j}(\boldsymbol{v})| + 2 \cdot \max_{i=1,2,3} \left| \frac{\partial \varphi}{\partial x_i} \right| \cdot \max_{i=1,2,3} |v_i| \\ &\leq |\epsilon_{i,j}(\boldsymbol{v})| + C_0 |\boldsymbol{v}| \\ \implies |\epsilon_{i,j}(\varphi \boldsymbol{v})|^2 &\leq |\epsilon_{i,j}(\boldsymbol{v})|^2 + C_0^2 |\boldsymbol{v}|^2 + 2C_0 |\epsilon_{i,j}(\boldsymbol{v})| \cdot |\boldsymbol{v}| . \end{aligned}$$

Thus,

$$\|\boldsymbol{\epsilon}(\varphi \boldsymbol{v})\|^2_{2,B(x_0,r)} \leq \|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{2,B(x_0,r)} + C_0^2 \|\boldsymbol{v}\|^2_{2,B(x_0,r)} + 2C_0 \int_{B(x_0,r)} |\epsilon_{i,j}(\boldsymbol{v})| \cdot |\boldsymbol{v}| \, \mathrm{d}x$$

$$\leq |\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{2,B(x_0,r)} + C_0^2\|\boldsymbol{v}\|^2_{2,B(x_0,r)} + 2C_0 \cdot \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)} \cdot \|\boldsymbol{v}\|_{2,B(x_0,r)}$$

$$\leq C^2 \left[\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)} + \|\boldsymbol{v}\|_{2,B(x_0,r)}\right]^2 ,$$

where $C = \max\{1, C_0\}$. Consequently,

$$\|\boldsymbol{\epsilon}(\varphi\boldsymbol{v})\|_{2,B(x_0,r)} \leq C\Big[\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{2,B(x_0,r)} + \|\boldsymbol{v}\|_{2,B(x_0,r)}\Big],$$

and the result follows. □

Now, in Cartesian coordinates, the vorticity $\boldsymbol{w} = \nabla \times \boldsymbol{u}$ is given by

$$\boldsymbol{w} = (w_1, w_2, w_3) = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_3}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right),$$

so that

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}.$$

Consequently, we have that

$$\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{2,B(0,\varepsilon)} = \sqrt{2}\|\boldsymbol{w}\|_{2,B(0,\varepsilon)} .$$

By Lemma 6.3, for every $t \in (-1, 0)$,

$$\|\nabla\boldsymbol{u}(t)\|_{2,B(0,\varepsilon/2)} \leq C\Big[\|\boldsymbol{w}(t)\|_{2,B(0,\varepsilon)} + \|\boldsymbol{u}(t)\|_{2,B(0,\varepsilon)}\Big]$$

$$\leq C,$$

since $\boldsymbol{w}$ is bounded and $\boldsymbol{u} \in L^\infty(-1, 0\,; L^2(B(0, 1)))$.

Hence,

$$\frac{1}{\varepsilon}\iint_{Q(z_0,\varepsilon/2)} |\nabla\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\varepsilon \to 0, \ \text{ as } \varepsilon \to 0.$$

By Theorem 5.8, the point $z_0 = 0$ is regular, a contradiction.

Notice that if $\Omega = \mathbb{R}^3$, then we have proved that every point is regular. Consequently, $\nabla^{k-1}\boldsymbol{u}$ is Hölder continuous for all $k \in \mathbb{N}$. One can then show that

$$\limsup_{t \nearrow T^*} \|\nabla\boldsymbol{u}\|_{2,\Omega} < \infty,$$

so that the strong solution can be extended beyond T^*.

6.3 Case II: $T_0 > T^*$

If $T_0 > T^*$, the situation is more complicated since the solution is not necessarily smooth for all $t < T_0$. We use the same argument to reduce to the canonical domain, and assumptions (6.1) and (6.3) still hold, but (6.2) is replaced by

$$\nabla^{k-1}\boldsymbol{u} \text{ is Hölder continuous in } Q_\varepsilon \text{ for every } \varepsilon \in (0,1] \text{ and } k \in \mathbb{N}. \tag{6.13}$$

Since $\boldsymbol{w}$ only involves terms that appear in $\nabla \boldsymbol{u}$, we know that $\boldsymbol{w}$, $\nabla \boldsymbol{w}$ and $\nabla^2 \boldsymbol{w}$ are bounded in Q_ε.

Assumption (6.11) is replaced by

$$\boldsymbol{b} \in L^\infty(-1,0\,;L^2(B(0,1))). \tag{6.14}$$

The same argument as before implies that η is bounded in Q_ε. Furthermore, we find that (after some algebra) one of the terms in $\nabla^2 \boldsymbol{w}$ is of the form

$$\left(\frac{1}{r}\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r^2}\right)(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \boldsymbol{e}_r \otimes \boldsymbol{e}_r) \otimes \boldsymbol{e}_\theta.$$

Note that

$$\frac{\partial \eta}{\partial r} = \frac{\partial}{\partial r}(w_\theta/r) = \frac{1}{r}\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r^2}.$$

Since

$$\frac{\partial U}{\partial x_i} = \frac{x_i}{r}\frac{\partial \eta}{\partial r},$$

and x_i/r is bounded in $Q_{5,\varepsilon}$, we conclude that U and $\nabla_5 U$ are bounded in $Q_{5,\varepsilon}$.

We define a weak solution to (6.6) by

$$\iint_{Q_5(0,1)} [-U\partial_t\phi + \phi\boldsymbol{b}\cdot\nabla_5 U + \nabla_5 U\cdot\nabla_5\phi]\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\,\phi \in C_0^\infty(Q_5(0,1)).$$

Now, we also have a maximum principle for functions in the class

$$L^\infty(-1,0\,;L^2(B(0,1))) \cap L^2(-1,0\,;W^{1,2}(B(0,1))),$$

see Ladyzhenskaya *et al.* [18] Theorem III.7.2. Assumption (6.14) and the fact that U and $\nabla_5 U$ are bounded allow us to apply this maximum principle. The same argument as before now implies that $z_0 = 0$ is regular.

Conclusion

Over the course of this dissertation, we have investigated some aspects of the mathematical theory of the Navier-Stokes equations. In Chapters 2 and 3 we introduced Leray-Hopf weak solutions and proved, using the Galerkin method, the existence of such solutions. In Chapter 4, we turned to the questions of uniqueness and regularity of solutions. We introduced the notion of a strong solution, and showed that a strong solution exists, at least for a short amount of time. We also described the importance of strong solutions by showing that they are unique in the class of Leray-Hopf weak solutions and are as smooth as the given data.

In Chapter 5, we introduced the theory of local regularity in order to analyse singularities in the flow. We introduced suitable weak solutions and presented the Caffarelli-Kohn-Nirenberg Theorem, which gives the estimate $\mathcal{P}^1(S) = 0$. Finally, in Chapter 6, we analysed regularity of solutions for axially symmetric flow without swirl, and showed how to use the vorticity equation and a clever change of variables to reduce the problem to a heat equation with drift.

The theory of the Navier-Stokes equations is vast, and we have only very briefly touched upon some of the interesting topics that one could study further. Though we have mainly focused on the three-dimensional problem, along the way we proved existence and uniqueness of weak solutions to both the linear and two-dimensional problems. I am now particularly interested in treating the equations as a dynamical system, and analysing the long-term behaviour of solutions.

One extension for the material in Chapter 6 is to see what happens when there is swirl. Another possible extension is to consider the problem of boundary regularity of solutions. In fact, one can show that there are no singular points on the boundary *except possibly at points on the axis of symmetry.* The question of regularity at these points is open.

The ultimate question of finite-time blow-up for the Navier-Stokes equations in three dimensions is still far from being answered. Since Leray's ground-breaking work of 1934, progress has been relatively slow. To settle this problem, it seems that some new, revolutionary ideas are needed.

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Appendices

A Notation

A.1 Domains and Sets

- $\mathbb{R}^n$ is n-dimensional Euclidean space.
- $\mathbb{N}$ is the set of positive integers.
- Ω is a bounded open set in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. If $T > 0$, then $\Omega_T \coloneqq \Omega \times (0, T)$, and $\overline{\Omega_T} = \overline{\Omega} \times [0, T]$.
- $B(x_0, r) = \{ x \in \mathbb{R}^3 \mid |x - x_0| < r \}$.
- $Q(z_0, r) \equiv Q(x_0, t_0, r) = B_r(x_0) \times (t_0 - r^2, t_0)$.
- $Q^*(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0 + r^2)$.
- $Q_\varepsilon = B(0, 1) \times [-1, -\varepsilon^2]$, with $\varepsilon \in (0, 1]$.
- $B_5(0, 1) = \{ x \in \mathbb{R}^5 \mid |x| < 1 \}$.
- $Q_{5,\varepsilon} = B_5(0, 1) \times (-1, -\varepsilon^2)$.

A.2 Function Spaces

- $C^k(\Omega_T) = \{ f : \Omega_T \to \mathbb{R} \mid f \text{ is } k\text{-times continuously differentiable in } x \}$.
- $C^2_1(\Omega_T) = \{ f : \Omega_T \to \mathbb{R} \mid f,\ \nabla f,\ \nabla^2 f,\ \partial_t f \in C(\Omega_T) \}$.
- $C^\infty(\Omega) = \big\{ f : \Omega \to \mathbb{R} \mid f \in C^k(\Omega)\ \forall k \in \mathbb{N} \big\}$.
- $C^\infty_0(\Omega) = \{ f \in C^\infty(\Omega) \mid \exists \text{ compact } K \subset \Omega \text{ such that } f = 0 \text{ in } \Omega \setminus K \}$.
- $C^\infty_{0,0}(\Omega) = \{ \boldsymbol{\varphi} \in C^\infty_0(\Omega) \mid \operatorname{div} \boldsymbol{\varphi} = 0 \}$.
- $L^p(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \|f\|_{p,\Omega} < \infty \}$.
- $W^{1,p}(\Omega) = \{ f \in L^p(\Omega) \mid \text{the weak derivative of } f \text{ exists and is in } L^p(\Omega) \}$.
- $W^{1,p}_0(\Omega) = \{ f \in W^{1,p}(\Omega) \mid \exists\, f_n \in C^\infty_0(\Omega) \text{ such that } \|f - f_n\|_{1,2,\Omega} \to 0 \}$.
- $H = H(\Omega) = \big\{ \boldsymbol{u} \in L^2(\Omega) \mid \exists\, \boldsymbol{\varphi}_n \in C^\infty_{0,0} \text{ such that } \|\boldsymbol{u} - \boldsymbol{\varphi}_n\|_{2,\Omega} \to 0 \big\}$.
- $V = V(\Omega) = \big\{ \boldsymbol{u} \in W^{1,2}(\Omega) \mid \exists\, \boldsymbol{\varphi}_n \in C^\infty_{0,0} \text{ such that } \|\nabla(\boldsymbol{u} - \boldsymbol{\varphi}_n)\|_{2,\Omega} \to 0 \big\}$.
- For $1 \le p < \infty$, $L^p(0, T; X) = \big\{ \boldsymbol{u} \in X \mid \|\boldsymbol{u}\|_{L^2(0,T;X)} < \infty \big\}$, where

$$\|\boldsymbol{u}\|_{L^p(0,T;X)} = \left(\int_\Omega \|\boldsymbol{u}(\cdot\,, t)\|^p_X \,\mathrm{d}t \right)^{1/p}.$$

- $L^\infty(0,T\,;X) = \left\{\, \boldsymbol{u} \in X \mid \|\boldsymbol{u}\|_{L^\infty(0,T;X)} < \infty \,\right\}$, where

$$\|\boldsymbol{u}\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot\,,t)\|_X .$$

B Calculus and Functional Analysis

B.1 Gronwall's Inequality

Let $f(t)$ be integrable and differentiable on $[0,T)$, $\beta(t)$ be integrable on $[0,T)$. Suppose that

$$f'(t) \le \beta(t) f(t), \quad \forall\, t \in [0,T).$$

Then

$$f(t) \le f(0) \exp\left(\int_0^t \beta(\tau)\,\mathrm{d}\tau \right), \quad \forall\, t \in [0,T).$$

B.2 Fubini-Tonelli Theorem

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is measurable, and suppose that either of the following repeated integrals exists and is finite:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,\mathrm{d}x \right)\,\mathrm{d}y, \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,\mathrm{d}y \right)\,\mathrm{d}x.$$

Then $f \in L^1(\mathbb{R}^2)$, and

$$\iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right)\,\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y \right)\,\mathrm{d}x.$$

B.3 Hahn-Banach Theorem

Let X be a normed space, M a proper subspace, and $f \in M'$. Then there exists $g \in X'$ such that

$$g|_M = f \quad \text{and} \quad \|g\|_{X'} = \|f\|_{M'} .$$

B.4 Riesz Representation Theorem

Let X be a Hilbert space with inner product $(\cdot\,,\cdot)$, X' be the dual space. The mapping $\Psi : X \to X'$ defined by $[\Psi(x)](y) = (y\,,x)$, $\forall\, y \in X$ is a conjugate-linear isometry. Thus, every $T \in X'$ has a unique representative x such that $T(y) = (y\,,x)$, $\forall\, y \in X$.

B.5 Banach-Alaoglu Theorem

Let X be a separable Banach space, $\{f_n\}_{n=1}^{\infty}$ a sequence of functions in X'. If $\sup_n \|f_n\|_{X'} < \infty$, then there exists $f \in X'$ and a weak-$\star$ convergent subsequence f_{n_k} such that

$$f_{n_k} \overset{\star}{\rightharpoonup} f$$

i.e. $f_{n_k}(x) \to f(x)$ for each $x \in X$. If X is reflexive, the corresponding result is the existence of a weakly convergent subsequence i.e. $g(f_{n_k}) \to g(f)$ for every $g \in X'$.

B.6 Arzelà-Ascoli Theorem

Let $\{f_n\}_{n=1}^{\infty} \subset C(\overline{\Omega})$. Then there exists a uniformly convergent subsequence if and only if:

1. Uniform boundedness: $\sup\limits_{n} \|f_n\|_{\infty,\Omega} < \infty$.
2. Uniform equicontinuity: $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that for any $x\,, y \in \Omega$ with $|x - y| < \delta$, we have that $|f_n(x) - f_n(y)| < \varepsilon$ for every n.

B.7 Fundamental lemma of Calculus of Variations

Let $f \in L^2(\Omega)$. Then

$$\int_{\Omega} f(x) \cdot \varphi(x)\,\mathrm{d}x = 0 \quad \forall\, \varphi \in C_0^{\infty}(\Omega) \implies f \equiv 0 \quad \text{a.e. in } \Omega.$$

B.8 Young's Inequality

Let $a\,, b \in \mathbb{R}$ be non-negative. Suppose $1 < p\,, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}\,.$$

B.9 Poincaré's Inequality

There exists a constant C independent of u such that

$$\|u\|_{2,\Omega} \leq C\,\|\nabla u\|_{2,\Omega}\,, \quad \forall\, u \in W_0^{1,2}(\Omega).$$

B.10 Sobolev Embedding Theorem

Let $1 \leq p \leq n$. Then $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for every $q \in [1, \frac{np}{n-p}]$.

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